Fundamental goal

the theory of schemes

Schemes, spectra, morphisms, sheaves of modules,
line bundles, projective morphisms (blowing up)
derived functors & cohomology

Philosophical perspective

Scheme perspective vs variety perspective.

variety = soln set to some eqns
static object

scheme = blueprint for getting such sets
a "schema"
more encoded by the eqns themselves
vs. the soln set

__ex:__ Solns to $x^2=0$ in $\mathbb{C}$ vs $\mathbb{C}[\varepsilon]/\varepsilon^2$

Basic mathematical meta-strategy is
"misuse of domain"

Given a set of eqns, implicitly need "base ring" R where coeffs live.

Can then consider solns in any comm. ring containing R - i.e. A an R-algebra.

Schema S is a rule for associating to R-algebras A $\rightsquigarrow$ S(A) "solns in A"

Note: these are functors

__Def__ an R-space is a functor... R-alg $\to$ Set

__Def__ Cat of R-spaces. (funct cat)

<u>example</u> $A \longmapsto A^n$ "affine space" $\mathbb{A}^n$

"no eqns"

or for any indexing set I $A \mapsto A^I = \prod^I A$

$\mathbb{A}^I$

note that for these spaces we have

$\mathbb{A}^I(A) = A^I$ is naturally in bijection w/

$\mathrm{Hom}_{\mathrm{R\text{-}alg}}(R[x_i]_{i\in I}, A)$

more generally, for any R-alg B, we get a space

$S_B(A) = \mathrm{Hom}_{\mathrm{R\text{-}alg}}(B, A)$

Going as a morphism

$(\mathrm{C.R.\text{-}Alg})^{op} \longrightarrow \mathrm{Spc}_R$

which spaces does this give us?

the original ones in consideration were defined by poly eqns.

So suppose we have eqns f_j, $j \in J$ in variables x_i, $i \in I$ $\quad f_j \in R[x_i]_{i \in I}$

and we set $S(A) = \{ (a_i) \in A^I \mid f_j(a_i) = 0 \text{ all } j \}$

then we find $S(A) = \mathrm{Hom}_{R\text{-alg}}\left(\dfrac{R[x_i]_{i \in I}}{(f_j)_{j \in J}}, A \right)$

So these are all of this form.

But – this is all of them since for any R alg B, we have a surj map

$$R[x_i]_{i \in I} \longrightarrow B \quad \text{some } I$$
$$(I = B \text{ for example})$$

and ker B gen. by some f_j's. so

$$B \simeq R[x_i] / (f_j) .$$

We call these spaces affine spaces or affine schemes.

But, as we have seen, it is often natural to consider more general objects

<u>such as:</u> "lines in a vector space"

or "conics passing through 2 pts"

or "line bundles on a variety"

While these are often relatable to eqns in some fixed variables, this generally doesn't properly capture what we want.

For example, $\mathbb{P}^n_{\mathbb{C}}$, lines in $\mathbb{C}^n$ is a "nonaffine variety"

so whatever $\mathbb{P}^n_R(A)$ means, it shouldn't be of the above form. (def is slightly subtle)

Therefore, it is natural to shift our perspective slightly to develop language to talk about these.

<u>What is Algebraic Geometry</u>?

This semester's answer:

AG is the study of moduli problems & their relationships to each other.

So – what is a moduli problem?

well – it is more of a perspective.

"enriched parameter space"

<u>examples</u>

- lines in v. space.
- describe solns to some poly eqns
- " line bundles on a surface X
- " maps $X \to Y$ between varieties
- " $X \xrightarrow{f} \mathbb{P}^1$ s.t. $f^{-1}(0) \cong X_0$ flat etc. some fixed X_0.

Crucial that we suffer from
"misuse of domain" in studying these

We will often want to interpret/enhance/define
moduli problems in terms of functors on Rings.

in fact, for some

<u>Def</u>: A moduli problem is a space.

<u>Human def</u>: . . . . . functor w/ nice description.

<u>ex</u>: $A \longmapsto \{$ iso. classes of A-modules $\}$

"Pic_R" $\longmapsto \{$ . . . . which are rk 1 projective $\}$

we would like to say a <u>schema</u> is a space which
"locally" looks like an affine space

But – chicken/egg problem – we don't know what "locally" means yet.

So, we need to start w/ "geometry" of those affine spaces ⟷ affine schemes.

Side comment:
we will define soon "schemes" and a fully faithful functor

$$
\begin{array}{ccc}
\mathrm{Sch}_R & \longrightarrow & \mathrm{Spc}_R \\
\cup & & \cup \\
\mathrm{AffSch}_R & \xrightarrow{\sim} & \mathrm{AffSpc}_R \\
 & \searrow^{\sim} & \text{Th eq} \\
\mathrm{Spec}\, R \longleftarrow A & & (\mathrm{ComAlg}_R)^{op}
\end{array}
\qquad
\begin{array}{c}
S_R \\
\uparrow \\
A
\end{array}
$$

So schemes aren't literally spaces, but give spaces and may be studied via spaces

So we begin with

$\boxed{\text{the geometry of Affine schemes}}$

Recall: if $X \subset \mathbb{A}^n_{\mathbb{C}}$ is an affine variety defined by some poly eqns $f_1, \dots, f_m$, we define

$$\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n] / (f_1, \dots, f_m)$$

to be its "ring of regular functions" which is independent of the presentation.

The fundamental insight of AG is that we can study X via the ring $\mathbb{C}[X]$

for example a pt of $X \ni p \longleftrightarrow$ max ideal $m_p \triangleleft \mathbb{C}[X]$

given as the kernel of

$$\begin{aligned} \mathbb{C}[X] &\xrightarrow{ev_p} \mathbb{C} \\ f &\longmapsto f(p) \end{aligned}$$

"<u>Def</u>:" An affine scheme is one that is determined by its ring of regular fcns.

Slightly more precisely, a scheme is a ringed space

that is, a topological space X w/ a rule $\mathcal{O}_X$

which associates to $U \subset X$ open a ring $\mathcal{O}_X(U)$

"ring of regular functions on U" together w/ restriction

maps $U \subset V$ $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ satisfying some

natural properties (next time will take it more carefully here)

for now, the salient feature is that a function being regular

is a "local" property — that is

if $f_i \in \mathcal{O}_X(U_i)$ U_i cover U

and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then $\exists! f \in \mathcal{O}_X(U)$

s.t. $f|_{U_i} = f_i$. ("Sheaf property")

For this reason, it is enough to define a ringed space

by giving the value of $\mathcal{O}_X$ on a basis of the

top for X.

The Nullstellensatz says that we have a corresp. of max'l ideals & points.

For a ring A, set $\mathrm{mSpec}\, A = \{\text{max'l ideals in } A\}$

So we have a bijection $\mathrm{mSpec}\, A \longleftrightarrow X$

For a general ring, can always pretend it is of the form $R[x \ldots]/I$. So by above it is "functions" on some soln set ...

We also have a notion of Zariski top – given $g \in \mathbb{C}[X]$ can set $D_g = \{p \in X \mid g(p) \neq 0\}$ and these form a basis for our topology on X

we notice that $p \in D_g \Leftrightarrow g(p) \neq 0 \Leftrightarrow$

$g \notin m_p \Leftrightarrow$

$m_p \in \mathrm{mSpec}\, \mathbb{C}[X][g^{-1}]$

Recall, if $S \subset A$ is a mult. set, we can define $A[S^{-1}]$ and $\mathrm{mSpec}\, A[S^{-1}] \to \{m \in \mathrm{mSpec} A \mid S \cap m = \varnothing\}$

(more generally same ad Spec A.)

we therefore find the inclusion

$$\mathrm{mSpec}\ \mathbb{C}[x][g^{-1}] \to \mathrm{mSpec}\ \mathbb{C}[x]$$

should be regarded as $D_g \to X$ and so get a topology on mSpec $\mathbb{C}(x)$ corresp to X.

We have seen that every ring gives a "space" and so it is natural to do this for a general ring A.

given A, define mSpec A w/ Zariski top given by f.g's.

but for more general A, this is too restrictive for points. (if not over an alg. closed field). (think $A = \mathbb{R}[x...]/...$) etc.

Rather, want pts of scheme corresp to A to corresp to homs $A \to L$ for <u>any field L</u>. (i.e. soln's in L)

we should say that if $\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "p'"'] & L \arrow[d] \\ & L' \end{tikzcd}$ commutes then P, P' are equivalent.

Note: these corresp. to prime ideals.

Given A, get a ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

via $\mathcal{O}_{\operatorname{Spec} A}(D_g) = A[g^{-1}]$

Takes work to check - this gives a well defined notion of regular function $\mathcal{O}_{\operatorname{Spec} A}(U)$.

Def An affine scheme is a ringed space (≃ to one) of the form $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

Def A scheme is a ringed space $(X, \mathcal{O}_X)$ s.t. $\exists$ a cover $\underset{\operatorname{Spec} A_i}{\overset{\cong}{U_i}} \subset X$ w/ $\mathcal{O}_X|_{U_i} \cong \mathcal{O}_{\operatorname{Spec} A_i}$

Warning Schemes $\hookrightarrow$ Ringed spaces is not full.
we need to restrict morphisms to get

connectomes ("locally graded spaces")