Recently

Groth. tops / sites (topoi = special kinds of sites)

Sheaves & stacks

We can consider our objects (sheaves / stacks) "in the large"
or "in the small"

ex: X a scheme, $\mathcal{O}_X$ a sheaf of rings on X

$\underline{Sch_S}$ cat of S-schemes $(X \to S)$

then this cat has a Groth top (Zariski top)

$\mathcal{O}$ a sheaf of rings on $\underline{Sch_S}$ i.e.

$$\mathcal{O} : \underline{Sch_S} \to \underline{Sets}$$
$$X \longmapsto \Gamma(\mathcal{O}_X)$$

note restrict to $U \subset X \to S$ $\qquad U \longmapsto \Gamma(\mathcal{O}_U) = \mathcal{O}_U(U) = \mathcal{O}_X(U)$

can check this is a sheaf because $\mathcal{O}_X$ is for any X.

Similarly:

Ex: X a scheme, $\underset{\sim}{Shv}_{X_{Zar}}$ the stack which associates to $U \subset X$ open the category $Shv(U)$

$\underline{Sch}_S$ can also consider $\underset{\sim}{Shv}_{\underline{Sch}_S}$ stack on $\underline{Sch}_S$

associates to $X \rightsquigarrow Shv(X)$.

Next step: Geometry of sheaves

$\underline{\square}$ Cat. obj.

$\underset{\sim}{\square}$ 2-Cat. obj.

Idea: we think of $\underline{Shv}(\underline{Sch}_S)$ as generalized schemes/spaces. "Zariski sheaves"

$\mathfrak{X}$ a sheaf on $\underline{Sch}_S$

The site of $\mathfrak{X}$: (also called $\mathfrak{X}$)

$$ob(\mathfrak{X}) = \{(U,x) \mid U \in \underline{Sch}_S,\ x \in \mathfrak{X}(U)\}$$

$(U,x) \xrightarrow{f} (V,y)$ is $f: U \to V$ s.t. $\mathfrak{X}(f)(y) = x$

and $\{(U_i,x_i) \to (U,x)\}$ cov if $\{U_i \to U\}$ is a cov.

Ex: Consider
$$\mathcal{O}: \underline{Sch}_S \longrightarrow Sets$$
$$U \longmapsto \Gamma(\mathcal{O}_U)$$

consider
$$(U,f) \xrightarrow{R} \{g \in \Gamma(\mathcal{O}_U) \mid g^2 = f\}$$
$U \in Sch_S$, $f \in \Gamma(\mathcal{O}_U)$

Show R is a sheaf on $\mathcal{O}$.

Ex: $(U, F) \longrightarrow$ Cat of $\Gamma(\mathcal{O}_U)$-mod
is not a stack.

Properties of schemes (& sheaves) & their morphisms

Local property of a scheme:
a scheme $(X, \mathcal{O}_X)$ is regular if $\forall p \in X$ the local ring $\mathcal{O}_{X,p}$ is regular (i.e. max'l ideal gen. by reg. seq.)

[R comm. ring, $r_1, \ldots, r_n$ is a reg. seq. for a R-mod M if r_1 is a nonzerodiv on M & if $r_2 \to r_n$ is a reg. seq for M/r_1M.]

[$r_1, \ldots, r_n$ is a reg. seq. if it is a reg seq. for R]

[(R, m) local is regular if m is gen. by a reg. sequence in R]

Properties of morphisms of schemes:
examples: open immersion, closed immersion

Def we say $\varphi : X \longrightarrow Y$ morphism of schemes is an <u>affine morphism</u> if $\forall$ open affine $U \subset Y$ ($U = \operatorname{Spec} A$)

$\varphi^{-1}(U) = X \times_Y U$ is affine.

scheme theoretic inv. image

Ex: Hartshorne II.5.17 φ affine $\iff \exists \{U_i\}$ cover of Y s.t. U_i affine & $\varphi^{-1}(U_i)$ affine.

Def We say $\varphi : X \longrightarrow Y$ is <u>quasicompact</u> if $\forall U \subset Y$ affine, $\varphi^{-1}(U)$ is a quasicompact top. space.

($\iff$ inv. image of q.comp is q.comp)

These properties also extend to spaces (i.e. $\underline{\mathrm{Shv}}(\underline{\mathrm{Sch}}_S)$)

Recall $\mathfrak{X} \in \underline{\mathrm{Shv}}(\underline{\mathrm{Sch}}_S)$ is representable if $\exists X \in \underline{\mathrm{Sch}}_S$ s.t. $\mathfrak{X} \simeq \mathrm{Hom}_{\underline{\mathrm{Sch}}_S}(-, X) = h_X$

Def A morphism of spaces $\varphi : \mathfrak{X} \longrightarrow \mathcal{Y}$ is <u>representable</u> (by schemes) if $\forall Z$, schemes & $\psi : h_Z \longrightarrow \mathcal{Y}$, the pullback $\mathfrak{X} \times_{\mathcal{Y}} h_Z$ is representable.

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\varphi"] \\ \mathcal{Z} \arrow[r, "\psi"'] & \mathcal{Y} \end{tikzcd}$$

$\mathcal{O} \in \underline{\mathrm{Shv}}(\underline{\mathrm{Schs}})$

$\sqrt{\mathcal{O}} \in \underline{\mathrm{Shv}}(\underline{\mathrm{Schs}})$ $\qquad \sqrt{\mathcal{O}}(U) = \{ (f,g) \in \Gamma(\mathcal{O}_U)^2 \mid f = g^2 \}$

$S = \mathrm{Spec}\, k$

$\sqrt{\mathcal{O}} \longrightarrow \mathcal{O}$

$(f,g) \longrightarrow f$

g

$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[dr] \arrow[drr, bend left, "g"] \arrow[ddr, bend right] & & \\ & \square \arrow[r] \arrow[d] & \sqrt{\mathcal{O}} \arrow[d, "\varphi"] \\ & \mathrm{Spec}\, A \arrow[r, "\psi \leftrightarrow a"'] & \mathcal{O} \end{tikzcd}$$

g^2

$\lambda : A \to B$ $\qquad \psi = ??$ $\qquad \psi \in \mathrm{Hom}_{\mathrm{Shv}}(h_{\mathrm{Spec}\, A}, \mathcal{O})$

$\sqrt{\mathcal{O}}(B) \leftrightarrow g = b \in B$ $\qquad = \mathrm{Hom}_{\mathrm{Fun}}(\quad)$

commutes: $\lambda(a) = b \leftrightarrow g$ $\qquad = \Gamma \mathcal{O}(\mathrm{Spec}\, A) = A$

map to $\mathrm{Spec}\, B$ $\square \longleftrightarrow (A \xrightarrow{\lambda} B)$ together w/ $b \in B$ "g" s.t. $b^2 = \lambda(a)$

$$\mathcal{D} = \operatorname{Spec} \frac{A[x]}{x^2-a}$$

i.e. $\operatorname{Spec} B \longrightarrow \mathcal{D}$

$$A \xrightarrow{\lambda} B$$

$$\Leftrightarrow \quad \frac{A[x]}{x^2-a} \longrightarrow B$$

$$x \longmapsto b$$

Def if $\mathfrak{X} \xrightarrow{\varphi} \mathcal{Y}$ is a rep. morphism of Zariski sheaves,

can now make sense of defns

1) φ is open (i.e. rep. $\mathfrak{X} \times_{\mathcal{Y}} Z \to Z$ are open imm.) Z scheme

2) φ is affine

3) q. compact etc...

not correct definition!

Def An algebraic ~~Zariski~~ Zar. space is a sheaf which (?) is a coequalizer of the form

$$U' \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} U$$

where U', U are representable and π_1, π_2 are open immersions.

equivalently

(??) $\mathfrak{X}$ is alg. if $\exists \{ U_i \to \mathfrak{X} \}$ ← "cover" open immersions

s.t. $\amalg U_i \to \mathfrak{X}$ surjective as sheaves &

s.t. $\amalg U_i \times_{\mathfrak{X}} \amalg U_i = \amalg U_i \times_{\mathfrak{X}} U_j$ has a

Zariski cover by affine sheaves also.

Prop (??) $\iff$ $\mathfrak{X}$ is representable by a scheme.