Proj Recap

Proj A (Assume A graded ring $A = \bigoplus_{n \in \mathbb{Z}} A_n$)

($\bigstar$ soon will assume A gen by A_1 as an A_0-algebra)

points of Proj A $\xleftrightarrow[\bigstar]{\text{bijection}}$ hom. primes $\setminus$ {irrelevant ideal $A_{>0}$}

closed subschemes of Proj A $\xleftarrow{\bigstar}$ hom ideals

$\widetilde{I} \triangleleft \mathcal{O}_{\mathrm{Proj}A} \longleftarrow I \triangleleft A$

$\emptyset \longleftarrow A_{>0} \triangleleft A$

$\widetilde{I \cdot A_{>0}}\big|_{D_+(f)} \longleftarrow I \cdot A_{>0}$

$= \widetilde{I}\big|_{D_+(f)}$

closed subschemes of Proj A $\xleftrightarrow{A \text{ Noetherian}}$ saturated hom ideals ?

$\mathrm{Sat}(I) = \{x \in A \mid A_n \cdot x \subset I \text{ some } n > 0\}$

$Z \hookrightarrow \mathrm{Proj}\, A \qquad A = k[x_0, \ldots, x_n]$

$p \in \qquad f \in A_d \qquad$ " $f(p)$ " $= 0$?

$Z \longleftrightarrow d$

$p \in D_+(x_i) \qquad f(p)$

$= \mathrm{Spec}\, k[x_j/x_i]_j \not\ni f$

<u>Prov. def</u> $f(p) = 0$ if $\qquad f/x_i^d$

$f/x_i^d(p) = 0$ all affine opens $D_+(x_i)$

<u>(Better?) def</u> let $\mathcal{O}(d) =$ sheaf of fns of deg d

$\mathcal{O}(d)(D_{x_i}) = \{ f/x_i^m \mid \deg f - m = d \}$

will see locally $\mathcal{O}(d) \simeq \mathcal{O}$ so can make sense of vanishing by identifying w/ $\mathcal{O}$.

i.e. $\mathcal{O}(d) =$ "locally free rk 1 sheaves" = "invertible sheaves"

= line bundles.

<u>Proj</u> Fix X a scheme

Can consider q. coh. sheaves of graded $\mathcal{O}_X$-algebras

$$\mathcal{A}/\mathcal{O}_X \quad \mathcal{A} = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathcal{A}_d$$

<u>$Gr\mathcal{O}_X$-Alg</u> is a stack on X (or on Sch/X)

i.e. $\{U_i \to U\}$ cover and $\mathcal{A}_i/\mathcal{O}_{U_i}$ as above

w/ isoms. $\mathcal{A}_i|_{U_i \cap U_j} \xrightarrow{\varphi_{ij}} \mathcal{A}_j|_{U_i \cap U_j}$

s.t. $\varphi_{ik}|_{U_i \cap U_j \cap U_k} = \varphi_{jk}|_{\cdots} \cdot \varphi_{ij}|_{\cdots}$

gives uniq. (up to uniq. iso.) $\mathcal{A}/\mathcal{O}_U$

Proj gives a morphism of stacks

$$\underset{\sim}{Gr\mathcal{O}_X\text{-}Alg}^{op} \xrightarrow{\text{Proj}} \underset{\sim}{RelSch}_X$$

$$\left.\begin{matrix} \mathcal{A}/U \\ U \to X \end{matrix}\right\} \longrightarrow \begin{matrix} \text{Proj}\,\mathcal{A}/U \\ U \to X \end{matrix}$$

actually only described this for U affine!

Because both sides are stacks (on Zar top / X)

to define a morphism, enough to define it locally

enough to define on affines

$$\mathcal{A}/\mathcal{O}_X \longrightarrow \underline{\mathrm{Proj}}\,\mathcal{A}/X$$

This is the def of relative proj.

$$\underline{\mathcal{O}_X\text{-Alg}} \xleftrightarrow{\underline{\mathrm{Spec}}} \underline{\mathrm{RelSch}}_X$$

Proj gets a lot: Silly ex: $A = A_0[x]$

$$\underline{\mathrm{Proj}}\,A_0[x] = \underline{\mathrm{Spec}}\,A_0$$

Q: what isn't a proj?

Reasonable candidates: many examples of non q.proj varieties are probably not.

Analogous constructions if A multigraded

$$A = \bigoplus_{d \in \mathbb{Z}^m} A_d$$

w/ more choices of irrel. ideals.

"Proj A" $\rightsquigarrow$ toric varieties.

What's a projective morphism?

Candidate defs: (Stacks 01W7)

too powerful

- ~~any morph of form $\operatorname{Proj} A \longrightarrow X$~~ ~~$A/\mathcal{O}_X$ graded.~~
- ~~any morph $f: Y \to X$ s.t. $\exists$ cov U_i of X s.t. $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are each as above.~~

Hartsh.

- a morph of the form $Y \underset{\text{closed immersion}}{\hookrightarrow} \mathbb{P}^n_X = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X \xrightarrow{\pi_2} X$
- a morph locally as above

$\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \dots, x_n]$

EGA II · of the form $\operatorname{Proj} A \to X$ where
§5.5
A f.g. as an alg by $(1 \in A_0)^{f}_{r} A_1$ over $\mathcal{O}_X$

· locally as above

if X, A Noeth. then locally EGA = locally Hartshorne.

if X q.proj$/_R$ then Hart = EGA II

<u>Def</u> X is projective (over R) if X is closed in $\operatorname{Proj} R[x_0, \dots, x_n]$

q.proj. .. if X is open in proj.

<u>Q:</u> If $Y \xrightarrow{f} X$ is EGA-projective and X Noeth, f finite type then is it EGA proj w/ A_d coh. for all d ?

Divisors & Line bundles

Weil Divisors Cartier Divisors Invertible sheaves

X Noeth scheme

<u>Weil divisors</u>: $\mathrm{Div}\, X$ = free ab gp gen. by closed integral subschemes of codim 1.

$= \{ \sum n_i [D_i] \mid D_i \subset X$ closed irred codim 1 $\}$

variety = irred reduced scheme = integral scheme

D_i = closed subvarieties of X of codim 1.

<u>X variety</u>, if $f \in K(X)$ rational function can talk about 0's & poles of f

given $p \in X$ codim 1 point (i.e. $\dim \overline{\{p\}} = \dim X - 1$)

to see "how much" f vanishes at p, consider image of f in $\mathcal{O}_{X,p}$ Noeth ring of dim 1

<u>simplifying assumption</u>: $\mathcal{O}_{X,p}$ regular ring.

<u>Fact</u>: Regular local rgs of dim 1 are discret val. rgs.

$\mathcal{O}_{X,\mathfrak{p}}$ has max'l ideal $\mathfrak{m}_{\mathfrak{p}}$, gen. by a single elmt $t \in \mathfrak{m}_{\mathfrak{p}}$

and $\mathcal{O}_{X,\mathfrak{p}} = \{ut^n \mid u \in \mathcal{O}_{X,\mathfrak{p}}^*, n \geq 0\}$
uniquely

and $\operatorname{frac} \mathcal{O}_{X,\mathfrak{p}} \underset{= k(X)}{=\!=} \{ut^n \mid u \in \mathcal{O}_{X,\mathfrak{p}}^*, n \in \mathbb{Z}\}$

can write $f = ut^n$ in $\mathcal{O}_{X,\mathfrak{p}}$ $\Rightarrow$ f vanishes at $\mathfrak{p}$ to order n

t = thing vanishes along $\overline{\mathfrak{p}}$

(or f has a pole if $n < 0$)

<u>**Define**</u> in above situation, define $v_{\mathfrak{p}}(f) = n$ if $f = ut^n$ in $\operatorname{frac} \mathcal{O}_{X,\mathfrak{p}}$

<u>**Def**</u> $(\operatorname{div} f) = \sum_{\mathfrak{p} \text{ codim } 1} v_{\mathfrak{p}}(f) [\overline{\mathfrak{p}}]$

<u>fact</u>: this is a <u>finite sum</u> (on an open set, f in a unit.

$f \in \operatorname{frac} A$ $\quad \operatorname{Spec} A \subset X$ open
$f = g/h$ $\quad \operatorname{Spec} A_h \subset X$ $\overset{B}{''}$

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<u>Cartier Divisor</u> : modeled on the idea of loc. cut out by single functions

Def A variety is an integral Noeth. scheme

all rings $\mathcal{O}_X(U)$ are integral domains

and X is irreducible

in this case, for $\operatorname{Spec} A \hookrightarrow X$ (variety) affine open,

$\operatorname{Spec} A$ dense in X and define $K(X) = \operatorname{frac} A$

function field of X

(doesn't depend on A)

$$\begin{array}{ccc} \operatorname{Spec} A & & \operatorname{Spec} B \\ & \operatorname{Spec} A \cap \operatorname{Spec} B & \\ \cup & & \cup \\ \operatorname{Spec} A_f & & \operatorname{Spec} B_g \end{array}$$