

Last time:

Concept of a universal δ funct

($A = \text{Sheaves of Ab. grps or } \mathcal{O}_X$
mods)

Abelian categories A, B

on X

Discussed the natur of δ funct

($B = \text{Ab. groups}$)

$T^0: A \rightarrow B$ which is a seq. of functs

$T^i: A \rightarrow B$ w/ "connecting maps"

way of associating a map $T^i(Z) \xrightarrow{d} T^{i+1}(X)$ from
exact seq $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in A

so that we get LES's.

Universal δ funct. Note: these exist whenever

T^i are "effaceable" for all $i > 0$

i.e. $\forall X \in A \exists E \in A$ and $X \hookrightarrow E$ inj.
($H^0 = 0$)

s.t. $T^i(E) = 0 \quad i > 0$.

In this case, T^0 determines all T^i 's.

and T^i 's are called the satellites of T^0 .

Q: In practice, what do you actually do?

How to make T^i 's?

If \exists "enough injectives" in \mathcal{A}
 then define $R^i F$ for $F: \mathcal{A} \rightarrow \mathcal{B}$
 (= T^i) left exact

by choosing an inj. resolution

$$0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

(exact seq. w/ I_i injective)

and set

$$R^i F(A) = H^i(0 \rightarrow F I_0 \rightarrow F I_1 \rightarrow F I_2 \rightarrow \dots)$$

Observe (by construction)

$$R^i F(I) = 0 \text{ if } I \text{ injective} \\ \& i > 0$$

Def \mathcal{A} has enough injectives if $\forall X \in \mathcal{A}$
 $\exists I \in \mathcal{A}$ injective s.t.
 $0 \rightarrow X \rightarrow I$ exact.

Remark: I injective means

$$\begin{array}{c} \uparrow \\ 0 \rightarrow A \rightarrow B \end{array}$$

(morally: how does inj. help?)

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

w/ A inj. \Rightarrow split

and any additive functor F
 preserves splittings.)

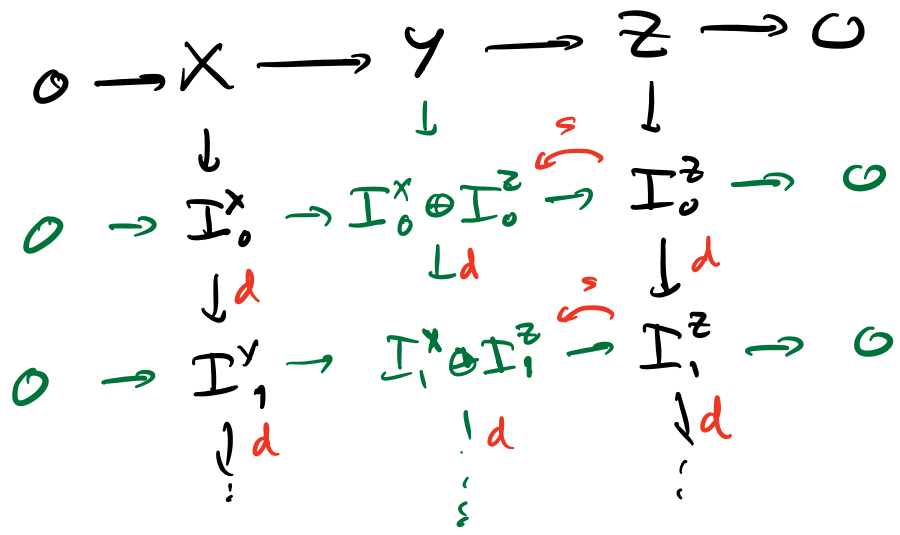
AbCat Refresher

If \mathcal{A} ab. cat. then have kernels

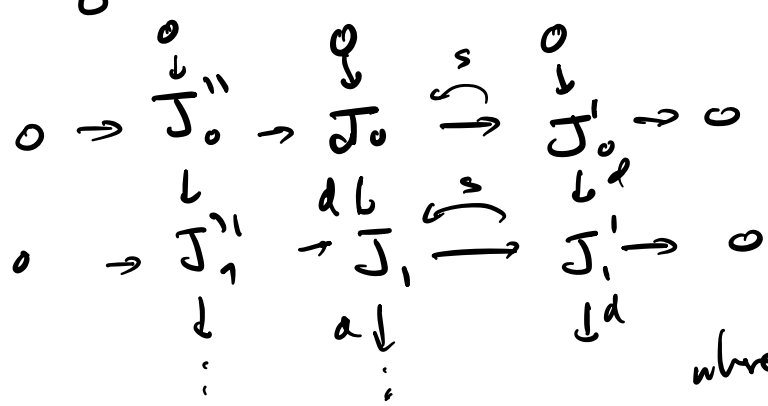
$$\begin{array}{c} X \xrightarrow{f} Y \\ \downarrow \text{ker } f \rightarrow X \xrightarrow{\quad 0 \quad} Y \\ \quad \quad \quad \downarrow f \end{array}$$

LFS's ?

Horseshoe Lemma gives away of fitly in middle of this diagram



more generally, given complexes



where each row is split exact

we get boundary maps
 $sd-ds : J_n' \rightarrow J_{n+1}''$
 and induce well defined maps

an exact seq $H^n(J_0') \rightarrow H^{n+1}(J_0'')$

in particular, start w/

$$I_0^x \rightarrow I_0^y \rightarrow I_0^z \rightarrow I_0^x[1]$$

termwise split exact seq of complexes

and so is $FI_0^x \rightarrow FI_0^y \rightarrow FI_0^z$

→
"an exact triangle"

$$\rightarrow FI_0^x[1]$$

Def If A an abelian cat, define $\mathcal{D}(A)$ "derived cat of A "
to be the cat w/ objects = complexes of objects in A

i.e. $\dots \rightarrow A_{-1} \rightarrow A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots$

morphisms eq. classes of maps of complexes

$$\begin{array}{ccccccc} A_{-1} & \rightarrow & A_0 & \rightarrow & A_1 & \rightarrow & \dots \\ \downarrow f_{-1} & & \downarrow f_0 & & \downarrow f_1 & & \\ B_{-1} & \rightarrow & B_0 & \rightarrow & B_1 & \rightarrow & \dots \end{array}$$

modulo "quasi-isom."

$f \sim g$ if induced maps $H^i(A_\bullet) \rightarrow H^i(B_\bullet)$ are same all i .

$\mathcal{D}(A)$ is a "triangulated" category
two extra bits of structure

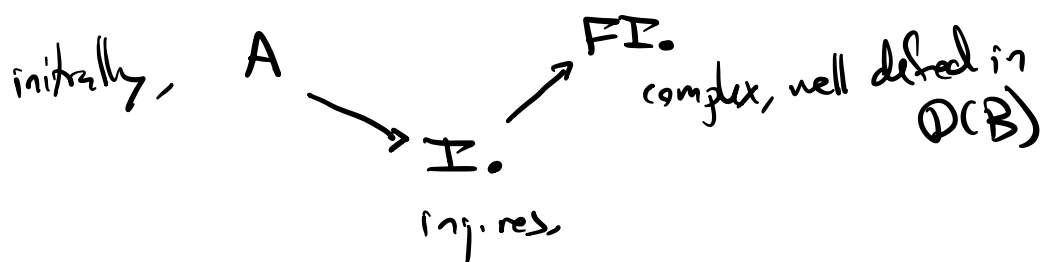
- Shift $C \mapsto C[1] \quad C[n]$
 $n \in \mathbb{Z}$
- A collection of "distinguished triangles"

$$A \rightarrow B \rightarrow C \rightarrow A[1]$$

$$B \rightarrow C \rightarrow A[1] \rightarrow B[1]$$

(stuff is not...
traverse split SES of complexes)
as above

From this perspective, the derived functor $R^i F$
can be interpreted as a functor $\mathcal{D}(A) \rightarrow \mathcal{D}(B)$



$$A \rightarrow \mathcal{D}(B)$$

given A_\bullet , can find a g.i.s.s $A_\bullet \xrightarrow[\text{g.i.s.}]{\simeq} \mathbb{I}_\bullet$
inj. complex

define $RF(A_\bullet) = \text{class of FI.}$

get a functor $\mathcal{D}(A_\bullet) \xrightarrow{RF} \mathcal{D}(B_\bullet)$

which is compatible w/ Δ 's in the sense that
 takes

$$A \rightarrow B \rightarrow C \rightarrow A[\iota]$$


$$RFA \rightarrow RFB \rightarrow RFC \rightarrow RFA[\iota]$$

Fact (Groth.) The category of sheaves of Ab. groups
 on any site (cat. w/ Groth. top) has enough injectives.

Hartshorne construction: in Ab. gp, divisible \Rightarrow injective.

$$A \hookrightarrow \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = \mathcal{I}(A)$$

$$\text{sheaf } \mathcal{F} \hookrightarrow \prod_{x \in X} (i_x)_* \mathcal{I}_x \hookrightarrow \prod_{x \in X} \mathcal{I}((i_x)_* \mathcal{F}_x)$$



In particular, we can define for any Groth. top. X
and any sheaf of Ab. grps \mathcal{F}

$$H^i(X, \mathcal{F}) = R^i \Gamma(X, \mathcal{F})$$

Almost no one ever has computed cohom successfully
in this way.

Computations in practice always come from
long exact sequence.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$\begin{aligned} H^n(X, \mathcal{F}) \\ \text{"} \\ R^n \Gamma(X, \mathcal{F}) \\ \pi: X \rightarrow \text{pt.} \\ \pi_* \mathcal{F} = \Gamma(\mathcal{F}) \end{aligned}$$

$$R^i (gf)_*$$

$$(R^p g_*) (R^q f_*) (\mathcal{F}) \Rightarrow R^{p+q} (gf)_* (\mathcal{F})$$

Example: Main important facts:

Sheaves vanish: if \mathcal{F} is a g -coh
 \mathcal{O}_X -mod, X an affe
scheme

$$\Rightarrow H^n(X, \mathcal{F}) = 0$$

more generally: if $f: X \rightarrow Y$ affe morphism $n > 0$.

$\Rightarrow R^i f_* \mathcal{F} = 0 \quad i > 0 \quad \mathcal{F} \text{ q.coh.}$
 "sheafification" of $u \mapsto H^i(p^{-1}u, \mathcal{F})$

\Rightarrow if f affine $X \rightarrow Y$ then
 $H^n(X, \mathcal{F}) = H^n(Y, f_* \mathcal{F})$

$H^p(Y, \underbrace{R^q f_* \mathcal{F}}) \Rightarrow H^{p+q}(X, \mathcal{F})$
 0 if $q > 0$
 $f_* \mathcal{F}$ if $q = 0$

If we have an open cover $\{U_i \rightarrow X\}$
 get a "combinatorial site"

objects: $U_i \quad U_{ij} \quad U_{ijk} \dots$
 $U_{ij}^k \quad U_{i|j|k}$

and inclusions as morphisms

i covers via $\{U_i \rightarrow X\}$ \cap this w/ covers open anlays.

"formal map of sites"

$$X \xrightarrow{f} \text{Comb}(U_\bullet)$$

$$H^p(\text{Comb } U_\bullet, R^0 \Omega_* \mathcal{F}) \Rightarrow H^{p+g}(X, \mathcal{F})$$

Can make a complex

$$\begin{array}{ccc} \prod_i R^0 \Omega_* \mathcal{F}(U_i) & \rightarrow & \prod_{i,j} R^0 \Omega_* \mathcal{F}(U_i \cap U_j) \\ \downarrow H^0(U_i, \mathcal{F}) & \searrow \Sigma(-1) \text{ res} & \downarrow \Sigma(-1) \\ & \underbrace{\hspace{10em}} & \\ & H^p(\) & \end{array}$$

if play cards right, and choose a cover s.t.
 $U_i, U_i \cap U_j, \dots$ all s.f.c.e.,
 \mathcal{F} s.coh

$$\Rightarrow H^z(U_i \cap \dots \cap U_j, \mathcal{F}) = 0 \quad z > 0$$

$$H^n(X, \mathcal{F}) = \check{H}^n(\{U_i\}, \mathcal{F})$$

$$\prod_i \Gamma(U_i, \mathcal{F}) \xrightarrow{\Sigma} \prod_{i,j} \Gamma(U_i \cap U_j, \mathcal{F}) \xrightarrow{\Sigma} \dots$$

Aside: if X is reasonable (locally contractible)
then can use above to compute Čech cohomology,
de Rham or Čech & singular.

Bott & Tu
