Last time:

Concept of a universal δ functor

Abelian categories A, B (A = Sheaves of Ab. gps or $\mathcal{O}_X$ mods) on X

Discussed the notion of a δ-functor (B = Ab. groups)

$T^\bullet : A \to B$ which is a seq. of functs

$T^i : A \to B$ w/ "connecting maps"

way of associating a map $T^i(Z) \xrightarrow{\delta} T^{i+1}(X)$ for every

exact seq $0 \to X \to Y \to Z \to 0$ in A

so that we get LES's.

Universal δ functr. Noted: these exist whenever

T^i are "effaceable" for all $i > 0$

i.e. $\forall X \in A$ $\exists E \in A$ and $X \hookrightarrow E$ inj. ($\ker = 0$)

s.t. $T^i(E) = 0$ $i > 0$.

In this case, T^0 determines all T^i's.

and T^i's are called the satellites of T^0.

Q: In practice, what do you actually do?

How to make T^i's?

If $\exists$ "enough injectives" in A then define R^iF for $F: A \to B$ left exact ($= T^i$)

by choosing an inj. resolution

$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$

(exact seq. w/ I_i injective)

and set

$R^iF(A) = H^i(0 \to FI_0 \to FI_1 \to FI_2 \to \cdots)$

Observe (by construction)

$R^iF(I) = 0$ if I injective & $i > 0$

Def A has enough injectives if $\forall X \in A$ $\exists I \in A$ injective s.t. $0 \to X \to I$ exact.

Reminder: I injective means

$$\begin{tikzcd} & I \\ 0 \arrow[r] & A \arrow[r] \arrow[u] & B \arrow[ul, dashed, "\exists"'] \end{tikzcd}$$

(morally: how does inj help? if $0 \to A \to B \to C \to 0$ w/ A inj. $\Rightarrow$ split and any additive functor F preserves splitness.)

AbCat Refresher

If A ab. cat. then have kernels

$$X \xrightarrow{f} Y$$

$$\ker f \to X \overset{0}{\underset{f}{\rightrightarrows}} Y$$

LES's?

Horseshoe Lemma gives a way of filling in middle of this diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_0^X \arrow[r] \arrow[d, "d"] & I_0^X \oplus I_0^Z \arrow[r] \arrow[d, "d"] & I_0^Z \arrow[r] \arrow[d, "d"] \arrow[l, bend right, "s"'] & 0 \\
0 \arrow[r] & I_1^Y \arrow[r] \arrow[d, "d"] & I_1^X \oplus I_1^Z \arrow[r] \arrow[d, "d"] & I_1^Z \arrow[r] \arrow[d, "d"] \arrow[l, bend right, "s"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

More generally, given complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J_0'' \arrow[r] \arrow[d] & J_0 \arrow[r] \arrow[d, "d"] & J_0' \arrow[r] \arrow[d, "d"] \arrow[l, bend right, "s"'] & 0 \\
0 \arrow[r] & J_1'' \arrow[r] \arrow[d] & J_1 \arrow[r] \arrow[d, "d"] & J_1' \arrow[r] \arrow[d, "d"] \arrow[l, bend right, "s"'] & 0 \\
 & \vdots & \vdots & &
\end{tikzcd}$$

where each row is split exact

we get boundary maps

$$sd - ds : J_n' \to J_{n+1}''$$

and induce well defined maps

on cohom $H^n(J_\bullet') \to H^{n+1}(J_\bullet'')$

in particular, start w/

$$I_\bullet^x \to I_\bullet^y \to I_\bullet^z \to I_\bullet^x[1]$$

termwise split exact sequence of complexes

and so is $FI_\bullet^x \to FI_\bullet^y \to FI_\bullet^z \to FI_\bullet^x[1]$

"an exact triangle"

<u>**Def**</u> If A an abelian cat, define $D(A)$ "derived cat of A"

to be the cat w/ objects = complexes of objects in A

i.e. $\cdots \to A_{-1} \to A_0 \to A_1 \to A_2 \to \cdots$

morphisms eq. classes of maps of complexes

$$\begin{array}{ccccccc} A_{-1} & \to & A_0 & \to & A_1 & \to & \cdots \\ \downarrow f_{-1} & & \downarrow f_0 & & \downarrow f_1 & & \\ B_{-1} & \to & B_0 & \to & B_1 & \to & \cdots \end{array}$$

modulo "quasi-isom."

$f \sim g$ if induced maps $H^i(A_\bullet) \to H^i(B_\bullet)$ are same all i.

$\mathbb{D}(\mathcal{A})$ is a "triangulated" category

two extra bit of structure

- Shift $C \longmapsto C[1]$ $\quad C[n]$, $n \in \mathbb{Z}$
- A collection of "distinguished triangles"

$$A \to B \to C \to A[1]$$

$$B \to C \to A[1] \to B[1]$$

(stuff q. iso to ... termwise split SES of complexes as above)

From this perspective, the derived functor R^iF can be interpreted as a functor $\mathbb{D}(\mathcal{A}) \to \mathbb{D}(\mathcal{B})$

initially,

$$A \to I_\bullet \to FI_\bullet$$

inj. res. $\quad$ complex, well defined in $\mathbb{D}(\mathcal{B})$

$$\mathcal{A} \to \mathbb{D}(\mathcal{B})$$

given $A_\bullet$, can find a q. iso $A_\bullet \underset{q.is}{\simeq} I_\bullet$ inj. complex define $RF(A_\bullet) =$ class of $FI_\bullet$

get a functor $D(\mathcal{A}) \xrightarrow[RF]{} D(\mathcal{B})$

which is compatible w/ Δ's in the sense that

takes $A \to B \to C \to A[1]$

$$RFA \to RFB \to RFC \to RFA[1]$$

<u>Fact</u> (Groth.) The category of sheaves of Ab. groups on any site (cat. w/ Groth. top) has enough injectives.

Hartshorne construction: in Ab. gp, divisible $\Rightarrow$ injective.

$$A \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = d(A)$$

sheaf $\mathcal{F} \hookrightarrow \prod_{x \in X} (i_x)_* \mathcal{F}_x \hookrightarrow \underbrace{\prod_{x \in X} d((i_x)_* \mathcal{F}_x)}_{\text{injective.}}$

In particular, we can define for any Groth. top. X and any sheaf of Ab. grps $\mathcal{F}$

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$$

Almost no one ever has computed cohom successfully in this way.

Computations in practice always come from Leray spectral sequence.

$$H^n(X, \mathcal{F}) = R^n\Gamma(X, \mathcal{F})$$

$\pi : X \to pt.$

$\pi_* \mathcal{F} = \Gamma(\mathcal{F})$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$R^i((gf)_*)$$

$$(R^p g_*)(R^q f_*)(\mathcal{F}) \Longrightarrow R^{p+q}(gf)_*(\mathcal{F})$$

<u>Example:</u> Main important facts:

Serre vanishing: if $\mathcal{F}$ is a q.coh $\mathcal{O}_X$-mod, X an affine scheme

$$\Longrightarrow H^n(X, \mathcal{F}) = 0 \quad n > 0.$$

more generally: if $f : X \to Y$ affine morphism

$\Rightarrow R^i f_* \mathcal{F} = 0 \quad i>0 \quad \mathcal{F}$ q.coh.

$\quad\quad\quad\quad \shortparallel$

sheafification of $U \mapsto H^i(f^{-1}U, \mathcal{F})$

$\Rightarrow$ if f affine $X \to Y$ then

$$H^n(X, \mathcal{F}) = H^n(Y, f_* \mathcal{F})$$

$$H^p(Y, \underbrace{R^q f_* \mathcal{F}}_{\substack{0 \text{ if } q>0 \\ f_* \mathcal{F} \text{ if } q=0}}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

If we have an open cover $\{U_i \to X\}$

get a "combinatorial site"

objects: $U_i \quad U_{ij} \quad U_{ijk} \ldots$

$\quad\quad\quad\quad U_{ij}^k \quad\quad U_i \cap U_j \cap U_k$

and inclusions as morphisms

$\{$ covers via $\{U_i \to X\}$ $\}$ $\cap$ this w/ covers given overlaps.

"formal map of sites"

$$X \xrightarrow{f} \mathrm{Comb}(U_\bullet)$$

$$H^p(\mathrm{Comb}\, U_\bullet, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

can make a complex complex

$$\underbrace{\prod_i \underset{H^q(U_i, \mathcal{F})}{R^q f_* \mathcal{F}(U_i)} \xrightarrow{\Sigma(-1)\,\mathrm{res}} \prod_{i,j} R^q f_* \mathcal{F}(U_i \cap U_j) \xrightarrow{\Sigma(-1)} \cdots}_{H^p(\ )}$$

if play cards right, and choose a cover s.t.

U_i, $U_i \cap U_j$, ... all affine,

$\mathcal{F}$ q.coh

$$\Rightarrow H^q(U_i \cap \dots \cap U_?, \mathcal{F}) = 0 \quad q > 0$$

$$H^n(X, \mathcal{F}) = \check{H}^n(\{U_i\}, \mathcal{F})$$

$$\prod_i \Gamma(U_i, \mathcal{F}) \xrightarrow{\Sigma(\,)} \prod_{i,j} \Gamma(U_i \cap U_j, \mathcal{F}) \xrightarrow{\Sigma(\,)} \cdots$$

Aside: if X is reasonable (locally contractible)

then can use above to compare Čech cohom i.e.

de Rham or Čech & singular.

Bott & Tu