Last time – started to think about "sheaves of top spaces"

(Philosophically: top spaces = enhanced sets)

Analogy w/ higher categories:

if X a top space

- points of X = objects
- paths in X : morphisms

2-category thinking: homotopies between paths = 2 morphisms etc.

simplicially: embddgs of Δ^2 simplicial 2-morph. thy.

Simplicial version of higher cats:

∞-cat assoc. to X = singular complex.

<u>Presheaves</u>: simplest choice

functor $\mathcal{C}^{op} \xrightarrow{\mathcal{X}} \underline{Top}$

site ↗ $\mathcal{C}$

$U \longmapsto \mathcal{X}(U)$ top space

what is the natural sheaf (stalk?) condition

Want: equiv. $\mathfrak{X}(U) \longrightarrow \prod_i \mathfrak{X}(U_i)$ s.t. compatibility

$\{U_i \to U\}$ cov

compatibility $\mathrm{Desc}(\{U_i\}, \mathfrak{X})$

(top space)

Sheaf/Stack condition

$$\mathfrak{X}(U) \xrightarrow[\text{h.eq.}]{\sim} \mathrm{Desc}(\{U_i\}, \mathfrak{X})$$

points of $\mathrm{Desc}(\{U_i\}, \mathfrak{X})$

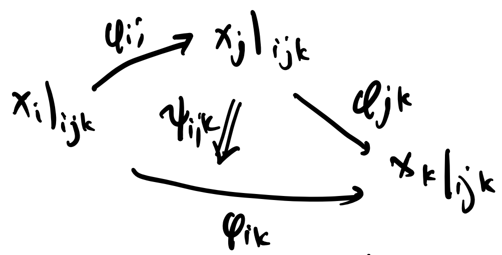
$(x_i) = x \in \prod_i \mathfrak{X}(U_i)$

Sheaves $\leftrightarrow \mathfrak{X}(U)$ discrete

$$x_i|_{U_i \cap U_j} \sim x_j|_{U_i \cap U_j}$$

$x_i|_{U_i \cap U_j} \in \mathfrak{X}(U_i \cap U_j)$, $x_j|_{U_i \cap U_j} \in \mathfrak{X}(U_i \cap U_j)$

φ_{ij} path in $\mathfrak{X}(U_i \cap U_j)$ from $x_i|_{ij}$ to $x_j|_{ij}$



Stacks $\leftrightarrow \mathfrak{X}(U)$ has contractible cnil. cams.

simplicial version (quasi-cat / ∞-cat lang.)

infinite chain of data = desc. data.

More compactly: $\mathrm{Desc}(\{U_i\}, \mathcal{X}) \simeq \mathrm{Tot}(\mathcal{X}(U_\bullet))$

let Δ = topological cosimplex

"Recall" cosimplicial object in a cat $\mathcal{C}$ is a diagram of the form $S_0 \rightrightarrows S_1 \rightrightarrows S_2 \cdots$

functor from finite ordered sets w/ weakly order preserving maps to some category.

$[n] = \{0, \dots, n\}$ $\quad$ $[0] \rightrightarrows [1] \rightrightarrows [2]$ $\quad$ skip maps d^i ? "coface"

i.e. cosimplicial obj in $\mathcal{C}$ $\quad$ $cS(\mathcal{C}) = \mathrm{Fun}(\underline{\mathrm{Fin}}, \mathcal{C})$

$\underline{\mathrm{Fin}} \ni$ objects $[n]$ $\quad$ $[k] \longrightarrow [n]$

analogously: a simplicial object is an object of $\quad$ $S(\mathcal{C}) = \mathrm{Fun}(\underline{\mathrm{Fin}}^{op}, \mathcal{C})$

Covers $\rightsquigarrow$ simplicial objects
Čech $\quad$ $\{U_i\}_{i \in I} \rightsquigarrow U_\bullet$

$$\mathcal{U} \dashleftarrow \left(U_0 \leftleftarrows U_1 \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; U_2 \right)$$

$$U_0 = \prod_{i \in I} U_i, \quad U_1 = \prod_{i,j \in I} U_i \times_{\mathcal{U}} U_j, \quad U_2 = \prod_{i,j,k \in I} U_{ijk}$$

$$U_i \longrightarrow U_i \times_{\mathcal{U}} U_i$$

$$U_i \times_{\mathcal{U}} U_j \xrightarrow{\Delta \times id} U_i \times_{\mathcal{U}} U_i \times_{\mathcal{U}} U_j$$

$$U_i \times_{\mathcal{U}} U_j \xrightarrow[id \times \Delta]{} U_i \times_{\mathcal{U}} U_j \times_{\mathcal{U}} U_j$$

apply $\mathcal{X} : \mathcal{C}^{op} \longrightarrow \underline{Top}$

get a cosimplicial object

$$\mathcal{X}(\mathcal{U}) \dashrightarrow \underbrace{\mathcal{X}(U_0) \rightrightarrows \mathcal{X}(U_1) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{X}(U_2)}_{\text{cosimplicial top space.}}$$

Try again:

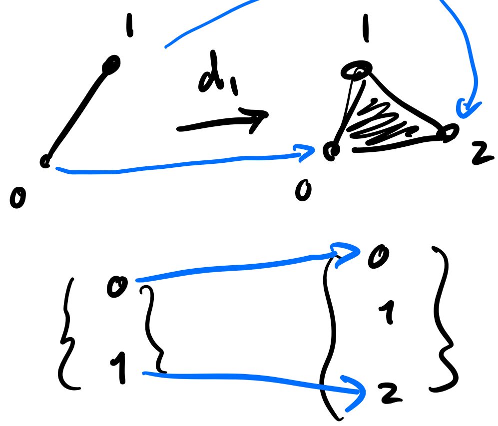
Define the cosimplicial top space Δ (the cosimplex)

to be: Δ_i = top i-simplex $= \{ x \in \mathbb{R}^{i+1} \mid \sum x_j = 1, \ x_j \geq 0 \}$

$$\Delta_i \xrightarrow{d_j} \Delta_{i+1} \qquad \Delta_i \longrightarrow \Delta_{i-1}$$

linear maps which extend maps

$$[i] \longrightarrow [i+1] \quad [i] \longrightarrow [i-1]$$



identify points on boundary of Δ_i's w/ finite sets.

if $\mathcal{Y}$ a cosimplicial space

define

$$\mathrm{Tot}(\mathcal{Y}) = \mathrm{Map}(\Delta, \mathcal{Y})$$

i.e. cont. maps

$$\Delta_n \to \mathcal{Y}_n$$

comp. w/ structure maps etc. d's s's.

i.e. $\Delta_0 \to \mathcal{Y}_0$, $\Delta_1 \to \mathcal{Y}_1$ (1_0, 1_1, ...)

pt y in $\mathcal{Y}_0$

path in $\mathcal{Y}_1$ connects $y|_0$ w/ $y|_1$

$$\text{points of } \mathrm{Map}(\Delta, \mathcal{X}(\mathcal{U}_\bullet)) = \overset{\text{pts of}}{\mathrm{Desc}}(\{U_i\}, \mathcal{X})$$

Define $\mathrm{Desc}(\{U_i\}, \mathcal{X}) \equiv \mathrm{Map}(\Delta, \mathcal{X}(\mathcal{U}_\bullet))$

$|\Delta| = \varinjlim \Delta_n$ is a top. space a CW complex.

<u>Def</u> $\mathcal{X}$ is a (homotopy) sheaf/stack (generally need hypercovers) if $\mathcal{X}(U) \xrightarrow{\sim} \mathrm{Desc}(\{U_i\}, \mathcal{X})$

is a homotopy equiv

Given a cat $X \rightsquigarrow$ space $|X| = |NX|$

$$C^{op} \longrightarrow \underline{Cat} \xrightarrow{N} \underline{Top}$$

(Above ~ practical intro to higher stacks along lines of Lurie / Toën)

Jardine approach $\quad C^{op} \longrightarrow \underline{Top}$

$\qquad\qquad\qquad\qquad \longrightarrow SSet$

weq: $\mathcal{F} \to \mathcal{G}$ w.eq. if stalks of presheaves of homotopy grps are $\cong$.

$$SPre(\mathcal{C})[weq^{-1}] = Shv(\mathcal{C})$$

Dugger/Hollander/Isaksen "Hypercovers & simp. presheaves"

$SPre(\mathcal{C})$

$Pre(\mathcal{C})$

define $\mathcal{F} \xrightarrow{f} \mathcal{G}$ w.eq.

if $\mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p$ iso all p

$$Pre(\mathcal{C})[weq^{-1}] = Shv(\mathcal{C})$$

$\mathrm{Pre}(\mathcal{C}) \longrightarrow \mathrm{Shv}(\mathcal{C})$

$\mathrm{Pre}(\mathcal{C}) \longrightarrow \mathrm{Shv}(\mathcal{C})$

localization as Homotopy category

$\mathrm{SPre}(\mathcal{C}) \longrightarrow \mathrm{SShv}(\mathcal{C})$

Comp of Ab grps $\longrightarrow$ Der. Cat   Hom. cat.

X

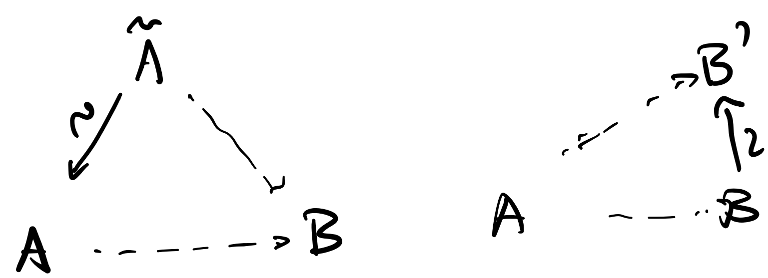
$\{U_i\}$ simp. presheaf of schemes / spaces

$\mathcal{F} \mapsto \Gamma(\mathcal{F})$

$H^i(\mathcal{F})$

Rings $\longrightarrow$ Ab grps or Rings

left derived functors



enlarge rings to simplicial rings

turns out poly rings are projective objects

Model cat thy: Dwyer Spalinsky

Akhil Mathew: notes on simplicial comm. gs

DAG-V + bits of higher topos thy