Things to potentially discuss (on Monday)

- Surfaces $\longrightarrow$ varieties / fields (alg closed) sm. proj.
  $\longrightarrow$ arithmetic situation: 2 dim'l Noeth schemes (regular, excellent)

birational classification (min'l model program)

$G \circlearrowright X$

G/P

- Survey of research directions
  - rat'l points, homogeneous varieties (under the action of Lin. alg gps)
  - Chow gp & int thy, moduli of v.bundles, gerbes, ...

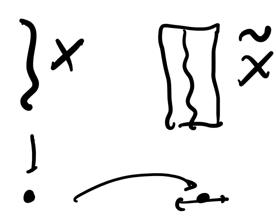
Basic problem of deformations of varieties/schemes.

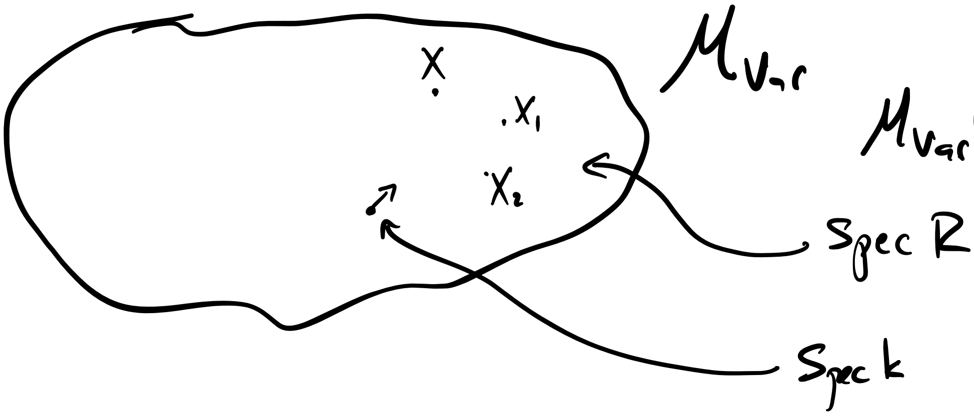
<u>problem</u>: $X/_k$ variety over k, want to describe all possible $\tilde{X}$ s.t.

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \end{array} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \tilde{X} \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \end{tikzcd}$$

"flat family"

Motivation:
Suppose want to parametrize all varieties





$\mathcal{M}_{Var}$

$$\mathcal{M}_{Var}(S) = \left\{ \begin{matrix} F \\ \downarrow \\ S \end{matrix} \right\}$$

<u>Def</u>: If Y is a k-scheme, $y \in Y$ a k-point i.e. $y : \operatorname{Spec} k \to Y$

then $T_y Y = \left\{ \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to Y \;\middle|\; \begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[dr, "y"'] & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[d] \\ & Y \end{tikzcd} \right\}$

(note: this is always a vector space over k)

$$\begin{tikzcd} & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[dr] & \\ \operatorname{Spec} k \arrow[ur] \arrow[rr, "y"] \arrow[dr] & & Y \\ & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[ur] & \cup \\ & & \operatorname{Spec} A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & k[\varepsilon]/\varepsilon^2 \arrow[d] \\ k[\varepsilon]/\varepsilon^2 \arrow[r] & k \end{tikzcd}$$

$$f(a) = f_0(a) + f_1(a)\varepsilon$$
$$\parallel$$
$$g(a) = g_0(a) + g_1(a)\varepsilon$$

$$(f+g)(a) = f_0(a) + \left(f_1(a) + g_1(a) \right)\varepsilon$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "y"'] & k[\varepsilon]/\varepsilon^2 \arrow[d] \\ & k \end{tikzcd}$$

$f(a) = y(a) + d(a)\varepsilon$

$f(ab) = f(a)f(b)$

$y(ab) + d(ab)\varepsilon$

$\shortparallel$

$y(a)y(b) + y(a)d(b)\varepsilon$
$\quad \shortparallel$
$\;y(ab) \qquad + y(b)d(a)\varepsilon$

$d(ab) = y(a)d(b)$
$\qquad + y(b)d(a)$

i.e. d is a y-derivation.

How do we calculate $\left\{ \begin{array}{c} \tilde{X} \\ \downarrow \text{ flat} \\ k[\varepsilon]/\varepsilon^2 \end{array} \middle| \text{ get } X \text{ when restricted to } k \right\}$?

$\tilde{X} \times_{\operatorname{Spec} k[\varepsilon]/\varepsilon^2} \operatorname{Spec} k = X$

$\operatorname{Def}(X)$ ("$= T_{[X]} \operatorname{Mod}_{Var}$")

tangent space of Mod_{Var} at $[X]$

$\operatorname{Mod}_{sch.}(S)$ = Category of flat S-schemes.

Suppose X is smooth want scheme $\tilde{X}$ over $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$

$\mathcal{O}_{\tilde{X}}(U)$ is a $k[\varepsilon]/\varepsilon^2$-algebra. s.t.

$\mathcal{O}_{\tilde{X}}(U) / \varepsilon\,\mathcal{O}_{\tilde{X}}(U) = \mathcal{O}_X(U)$

some $\varepsilon^2 = 0 \rightsquigarrow X \hookrightarrow \tilde{X}$ closed subscheme cut out by ideal sheaf $\varepsilon \mathcal{O}_{\tilde{X}}$

and as $\varepsilon^2 = 0$, same reduced scheme structure

$\mathrm{Spec}\, R = \mathrm{Spec}\, R/\sqrt{R}$

(i.e. $X \simeq \tilde{X}$ as top spaces)

So $\tilde{X}$ given by new sheaf rings on same top space.

<u>local case:</u> $X = \mathrm{Spec}\, A$ where A/k smooth algebra.

$\tilde{X}$ given by some $\mathrm{Spec}\, A' \hookleftarrow \mathrm{Spec}\, A$

where $\varepsilon A' = I$ $\quad A' \twoheadrightarrow A$

$$A'/I \simeq A$$

$$\begin{tikzcd}
\mathrm{Spec}\, A \arrow[r, equal] \arrow[d, "\text{quot. by square 0 ideal}"'] & \mathrm{Spec}\, A \arrow[d, "\text{smooth}"] \\
\mathrm{Spec}\, A' \arrow[r] \arrow[ur, dotted] & \mathrm{Spec}\, k
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \underset{\varepsilon A'}{\overset{\parallel}{I}} \arrow[r] & A' \arrow[r] & A \arrow[r] \arrow[l, bend left] & 0
\end{tikzcd}$$

$I \triangleleft A'$

$\Uparrow$

flat

$A' \otimes_{k[\varepsilon]/}$

$\downarrow$

$0 \to k \xrightarrow{\varepsilon} k[\varepsilon]/\varepsilon^2 \to k \to 0 \longrightarrow A'/\varepsilon A' \xrightarrow{\cdot\varepsilon} A' \to A'/\varepsilon A'$

$k \cong k[\varepsilon]/\varepsilon \quad A'/\varepsilon A' = I \quad A'/\varepsilon A' = A$

$\otimes_{k[\varepsilon]/\varepsilon^2} A'$

$$A' \simeq A \oplus I$$

$$(a \oplus x)(b \oplus y)$$
$$= ab \oplus (xb + ay)$$

$$\mathrm{Def}_{k\text{-alg}}(A) = \{A \oplus I\} = \{A \oplus A\varepsilon\}$$

$\mathrm{Def}(X)$ choose cover $\{U_i\}$ of X

if $\tilde{X}$ is a deformation of X then

$$\mathcal{O}_{\tilde{X}}(U_i) = \mathcal{O}_X(U_i) \oplus \mathcal{O}_X(U_i)\varepsilon$$
$$\mathcal{O}_X(U_i)[\varepsilon]/\varepsilon^2$$

i.e. $\tilde{X}$ covered by $\mathcal{O}_X(U_i)[\varepsilon]/\varepsilon^2$'s over the U_i's.

& isos $\mathcal{O}_X(U_i)[\varepsilon]/\varepsilon^2\Big|_{U_i \cap U_j} \underset{\varphi_{ij}}{\xrightarrow{\sim}} \mathcal{O}_X(U_j)[\ldots]\Big|_{\ldots}$

$= \mathcal{O}_X(U_i \cap U_j)[\varepsilon]/\varepsilon^2$

$$\varphi_{ij} : \underset{\overset{\shortparallel}{A_{ij}}}{\mathcal{O}_X(U_i \cap U_j)}[\varepsilon]/_{\varepsilon^2} \circlearrowleft$$

s.t. $\varphi_{ij} \otimes k[\varepsilon]/_{\varepsilon} : \mathcal{O}_X(U_i \cap U_j) \circlearrowleft \mathrm{id}.$

$$\mathrm{Hom}(A_{ij}[\varepsilon]/_{\varepsilon^2}, A_{ij}[\varepsilon]/_{\varepsilon^2})$$

$k[\varepsilon]/_{\varepsilon^2}$ $\quad \varepsilon \longmapsto \varepsilon$

$$\shortparallel$$

$$\mathrm{Hom}_k(A_{ij}, A_{ij}[\varepsilon]/_{\varepsilon^2})$$

$$\in$$

actually want φ_{ij} s.t.

$$\begin{tikzcd} A_{ij} \arrow[r, "\varphi_{ij}"] \arrow[dr, equal, "id"'] & A_{ij}[\varepsilon]/_{\varepsilon^2} \arrow[d] \\ & A_{ij} \end{tikzcd}$$

$$\varphi_{ij}(a) = a + d_{ij}(a)\varepsilon$$

i.e. φ_{ij} given by a derivation $A_{ij} \to A_{ij}$

recall, have A_{ij}-module $\Omega_{A_{ij}}$ s.t.

$$\mathrm{Hom}(\Omega_{A_{ij}}, M) = \mathrm{Der}(A_{ij}, M)$$

$$\mathrm{Der}(A_{ij}, A_{ij}) = \mathrm{Hom}(\Omega_{A_{ij}}, A_{ij}) = \Omega^*_{A_{ij}} = T_{A_{ij}}$$

i.e. can consider $\varphi_{ij} \in T_X(U_i \cap U_j)$

claim: composition $\varphi_{jk}\varphi_{ij}$ $\underset{\substack{\wedge \\ \mathcal{O}_X(U_i \cap U_j \cap U_k)}}{\overset{\text{comp in}}{\mathrm{Hom}(A_{ijk}[\varepsilon]/\varepsilon^2, A_{ijk}[\varepsilon]/\varepsilon^2)}}$

$\underbrace{}_{\text{adding in tangent sh.}}$

$\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ defines a Čech cohomology class in $H^1(X, T_X)$

<u>Conclusion</u>: Iso classes of objects in $\mathrm{Def}(X)$ are in bijection w/ $H^1(X, T_X)$.

(i.e. sheaf cohomology)

<u>observation</u>: if X is affine (thm of Serre $\Rightarrow$ $H^i(X, \underset{\text{q.coh}}{-}) = 0$, $i > 0$)

$\Rightarrow$ no nontrivial deformations for affines.

$X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2$

$$A \oplus A\varepsilon = A[\varepsilon]/\varepsilon^2 = A \otimes_k k[\varepsilon]/\varepsilon^2$$

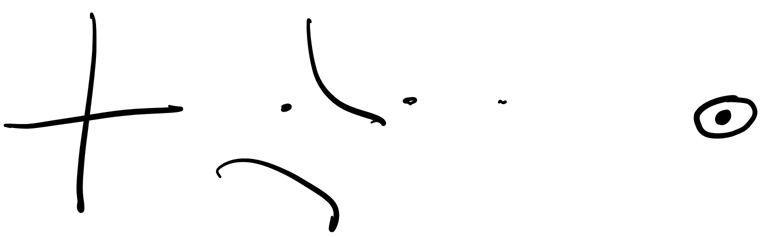
$\mathcal{M}_{1,1}$ Elliptic curves \ pt.

affine curves

if C, C' genus g curves $/\mathbb{C}$, $p \in C$ $p' \in C'$

$C \setminus \{p\} \simeq C' \setminus \{p'\}$

$$\frac{k[x,y]}{xy=a} \simeq k[x,x^{-1}]$$



$\mathfrak{X}$ "space or stack or higher stack"
parametrizing forms of varieties
line bundles on a fixed scheme ...

given $\mathfrak{X} : k\text{-alg} \longrightarrow$ Sets, Cats, S.Sets, Top...

$$A \longmapsto x \in \mathfrak{X}(A)$$

$$\tilde{x} \in \mathfrak{X}(A') \text{ s.t. } \tilde{x}|_A = x$$

$$I \to A' \to A$$

I nilpotent ideal

$$\mathrm{Def}_x(A') = \left\{ \tilde{x} \in \mathfrak{X}(A') \; ; \; \text{isom } \tilde{x}|_A \xrightarrow{\sim} x \right\}$$

$$k[\varepsilon]/\varepsilon^{n+1} \to k$$ nth order deformation.

$I \lhd A'$ $\quad A'/I = A$

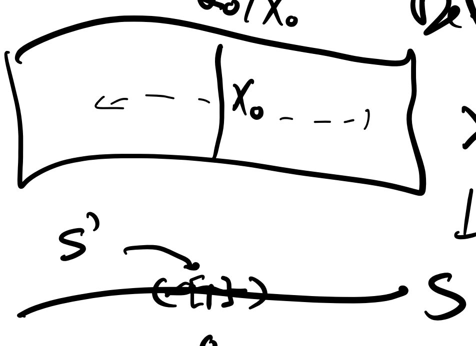
A' is I-adically complete.

$\mathrm{Def}_x(A')$ = formal deformations.

$\mathrm{Pic}_{X/S}$

$\downarrow$

S



$X_0 \hookrightarrow \mathbb{P}^N$

$\lim_n \mathcal{O}_S/m_S^n$ $\quad \hat{\mathcal{O}}_{S,s_0}$

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & \mathrm{Pic}_{X/S} \arrow[d] \\ & S \end{tikzcd} \qquad \begin{tikzcd} X_Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

$\mathrm{Hom}(Y, \mathrm{Pic}_{X/S}) = \mathrm{InvShv}(X_Y)$ a category.

$Y \mapsto$ Stack.

$\mathrm{Hom}(k, \mathrm{Pic}_{X/S})$

$k \longrightarrow s_0 \longrightarrow S$