Last time: Affine schemes:

Defined $\operatorname{Spec} R$ as a set & top. space.

Said a bit about sheaves.

- Def of sheaf
- Sheafification of a presheaf
- notions of injectivity/surjectivity for sheaves vs. presheaves.

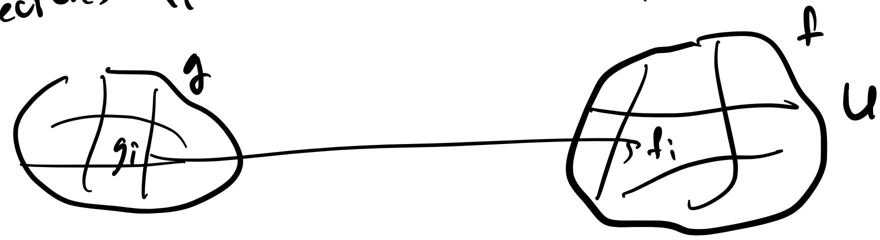
Def A morphism of presheaves $\varphi: \mathcal{F} \to \mathcal{G}$ is injective/surj iff $\forall\, U \subset X$ open $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ is inj/surj.

Def A morphism of sheaves $\varphi: \mathcal{F} \to \mathcal{G}$ is inj/surj

stalk surj $\Big\{$ iff $\forall\, x \in X$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is inj/surj.
(stalks)

Example statement: A morphism $\varphi: \mathcal{F} \to \mathcal{G}$ is surj

loc. surj. $\Big\{$ iff $\forall\, U$, $g \in \mathcal{G}(U)$ $\exists$ cover U_i of U and sections $f_i \in \mathcal{F}(U_i)$ s.t. $g|_{U_i} = \varphi(U_i)(f_i)$



Proof: loc. surj $\Longrightarrow$ stalk surj.

Suppose φ is loc. surj. WTS stalk surj. So let $g \in \mathcal{G}_x$ WTS

$\exists\, f \in \mathcal{F}_x$ s.t. $\varphi_x(f) = g$.

$\mathcal{G}_x = \varinjlim_{U \ni x} \mathcal{G}(U)$. So $\exists\, U \ni x$ and $\tilde{g} \in \mathcal{G}(U)$ s.t. $g = $ im of $\tilde{g}$ under $\mathcal{G}(U) \to \mathcal{G}_x$

loc. surj $\Rightarrow \exists$ cover U_i of U and $f_i \in \mathcal{F}(U_i)$ s.t. $\tilde{g}|_{U_i} = \varphi(U_i)(f_i)$

$x \in U_i$ some i.

$$\begin{tikzcd}
 & \mathcal{G}(U) \ni \tilde{g} \arrow[d] \\
\mathcal{F}(U_i) \ni f_i \arrow[r, "\varphi(U_i)"] \arrow[d] & \mathcal{G}(U_i) \ni \tilde{g}|_{U_i} \arrow[d] \\
\mathcal{F}_x \ni [f_i] \arrow[r] & \mathcal{G}_x \ni g
\end{tikzcd}$$

diagram $\Rightarrow g = $ im of $[f_i] \in \mathcal{F}_x$

Conversely, assume φ is stalk surjective. WTS loc. surjective.

Choose $g \in \mathcal{G}(U)$. Know $\exists\, f_x \in \mathcal{F}_x$ $f_x \longmapsto g_x$ all x.

each $f_x = $ im of some $\tilde{f}_x \in \mathcal{F}(U_x)$ some $U_x \ni x$ open

$$\begin{tikzcd} \tilde{f}_x \arrow[r, mapsto] & \varphi(U_x)(\tilde{f}_x) \arrow[dr] & & g|_{U_x} \arrow[dl] \\ & & g_x \in \mathcal{G}_x & \end{tikzcd}$$

Prop of direct limits: if two things = in lim. they are = at a finite stage.

i.e. $\exists\, V_x \subset U_x$ s.t. $\varphi(U_x)(\tilde{f}_x)\big|_{V_x} = g|_{V_x}$

$$\begin{tikzcd} \mathcal{F}(U_x) \arrow[r, "\varphi(U_x)"] \arrow[d] & \mathcal{G}(U_x) \arrow[d] \\ \mathcal{F}(V_x) \arrow[r, "\varphi(V_x)"'] & \mathcal{G}(V_x) \end{tikzcd}$$

$\varphi(V_x)(\tilde{f}_x|_{V_x})$

i.e. call $\tilde{\tilde{f}}_x = \tilde{f}_x|_{V_x}$

have an open cover V_x and $\tilde{\tilde{f}}_x \in \mathcal{F}(V_x)$ s.t. $\tilde{\tilde{f}}_x \longmapsto g|_{V_x}$. $\square$

<u>Today</u>: structure sheaf (Spec R as a locally ringed space)

Suppose $\mathcal{B}$ is a basis of open sets for a top on X.

$\mathcal{F}$ a Sheaf: $\mathcal{F}: \mathrm{Open}(X)^{op} \to \mathcal{C}$

s.t. $\forall\, U_i$ covs of U

$$\mathcal{F}(U) = \mathrm{eq}\left(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)\right)$$

$\mathcal{F}$ is a $\mathcal{B}$-sheaf: $\mathcal{F} : \mathcal{B}^{op} \longrightarrow \mathcal{C}$

subcat of $\mathrm{Open}(X)$ consisting of elmts of $\mathcal{B}$.

$\forall \{U_i\}$ covy U in $\mathcal{B}$, and $\{V^{ij}_k\}$ covy $U_i \cap U_j$

$\{U_i\} \in \mathcal{B}$, $U \in \mathcal{B}$, $V^{ij}_k \in \mathcal{B}$

we have $\mathcal{F}(U) = \mathrm{eq}\left(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(V^{ij}_k) \right)$

morphism of $\mathcal{B}$-sheaves = nat. transformations as before.

We have a functor $\mathcal{C}\text{-}\mathrm{Shv}(X) \longrightarrow \mathcal{C}\text{-}\mathcal{B}\mathrm{Shv}(X)$

$$\left[\mathcal{F} : \mathrm{Open}(X)^{op} \to \mathcal{C} \right] \longrightarrow \left[\mathcal{F}|_{\mathcal{B}^{op}} : \mathcal{B}^{op} \to \mathcal{C} \right]$$

In fact:

__Prop:__ The above is an __isomorphism__ of categories.

Use this to define the sheaf $\mathcal{O}_X$ $X = \mathrm{Spec}\, R$

"structure sheaf of regular functions on X"

__Def__ $\mathcal{O}_X(X_f) = R_f$

what does it mean for X_{f_i} to cover X?

$\bigcup X_{f_i} = X = \operatorname{Spec} R$ means $\forall \mathfrak{p}$ prime,

$\mathfrak{p} \in X_{f_i}$ some i $\quad X_{f_i} = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$

covering $\Leftrightarrow$ $\forall \mathfrak{p}$ prime, $\exists i$, $f_i \notin \mathfrak{p}$.

$\Leftrightarrow$ $(f_i)_{i \in I} \not\subset \mathfrak{p}$ any prime $\mathfrak{p}$.

$\Rightarrow$ not contained in any max'l $\Rightarrow$ unit ideal.

conversely $(f_i)_{i \in I} = R \Rightarrow \ldots$ cover.

$(f_i)_{i \in I} \subset \mathfrak{p} \Leftrightarrow \sqrt{(f_i)_{i \in I}} \subset \mathfrak{p}$

$\Leftrightarrow \sqrt{(f_i^N)} \subset \mathfrak{p}$

$\Leftrightarrow (f_i^N) \subset \ldots$

<u>Cor</u>: X is quasicompact: covers have finite subcovers.

if U_i covers X choose V_{ij} basic covering U_i

$V_{ij} = X_{f_{ij}}$ $\quad (f_{ij}) = R \Rightarrow 1 = \sum_{ij \in K} a_{ij} f_{ij}$

K finite.

$\Rightarrow (f_{ij})_{ij \in K}$ also $= R \Rightarrow X_{f_{ij}}$ $ij \in K$ cov.

$\Rightarrow U_i$ st $\exists j$ $ij \in K$ cov.

<u>Claim</u>: $\mathcal{O}_X$ is a $\mathcal{B}$ shef, $\mathcal{B} = \{X_f\}$ basic opens.

WTS if $U \in \mathcal{B}$ $\underset{\cap \mathcal{B}}{U_i}$ cov U $\underset{\cap \mathcal{B}}{V_k^{ij}}$ cov $U_i \cap U_j$

then $\mathcal{O}_X(U) = eq(\prod \rightrightarrows \prod \quad)$

i.e. i) $\mathcal{O}_X(U) \hookrightarrow \prod \mathcal{O}_X(U_i)$

ii) given $s_i \in \mathcal{O}_X(U_i)$ s.t. $s_i|_{V_k^{ij}} = s_j|_{V_k^{ij}}$ all k

then $\exists\, s \in \mathcal{O}_X(U)$ s.t. $s|_{U_i} = s_i$

i: $U = X_f$ $\qquad U_i = X_{g_i}$ $\qquad X_{g_i} \subset X_f$

$X_{g_i} \subset X_f$ $\quad \forall \mathfrak{p},\ \mathfrak{p} \in X_{g_i} \Rightarrow \mathfrak{p} \in X_f$

$g_i \notin \mathfrak{p} \Rightarrow f \notin \mathfrak{p}$

$f \in \mathfrak{p} \Rightarrow g_i \in \mathfrak{p}$

i.e. $g_i \in \bigcap_{\mathfrak{p} \ni f} \mathfrak{p} = \sqrt{(f)}$ $\qquad$ $X_{g_i} \subset X_f$ means $g_i \in \sqrt{(f)}$

X_{g_i} cov X_f means

$\forall \mathfrak{p}, \; f \notin \mathfrak{p} \Rightarrow \exists i$ s.t. $g_i \notin \mathfrak{p}$

$\forall \mathfrak{p}, \forall i, \; g_i \in \mathfrak{p} \Rightarrow f \in \mathfrak{p}$

i.e. $f \in \sqrt{(g_i)} \Rightarrow \sqrt{(\underset{\uparrow}{g_i})}$ not proper in R_f

$\Rightarrow (g_i)$ not proper in R_f $\qquad 1 = \sum a_i g_i$ in R_f

in fact, can check this is iff.

i.e. $\qquad X_{g_i}$ cov $X_f \Leftrightarrow (g_i) = R_f$

X_{f_i} cov $X \Leftrightarrow (f_i) = R$.

Suppose $s, s' \in R_f$ s.t. s, s' same img in R_{g_i} all i

WTS $s = s'$. $\quad$ consider $t = s - s'$ $\quad t \longmapsto 0$ in R_{g_i}

$\Rightarrow$ choose N s.t. $f^N t = $ im. of $\tilde{t} \in R$

$$t \longmapsto 0 \text{ in } R_{g_i} \Rightarrow \tilde{t} \longmapsto 0 \text{ in } R_{g_i}$$

$$\tilde{t} \to f^M t$$

$$\ker(R \longrightarrow R_{g_i}) = \{ r \in R \mid g_i^M r = 0 \text{ some } M. \}$$

$\tilde{t} g_i^M = 0$ some M. all i.

but $(g_i) = 1$ in $R_f \Rightarrow (g_i^M) = 1$ in R_f

$$1 = \sum g_i^M \frac{r_i}{f^{N'}} \text{ in } R_f$$

$$f^{N'} = \sum g_i^M r_i \text{ in } R_f$$

$$f^{N''} = f^{N'} \sum g_i^M r_i \text{ in } R$$

$$\tilde{t} f^{N''} = \tilde{t} f^{N'} \sum g_i^M r_i$$

$$= f^{N'} \sum \tilde{t} g_i^M r_i = 0$$

$$\Rightarrow \tilde{t}/1 = 0 \text{ in } R_f \Rightarrow t = 0 \text{ in } R_f.$$

$$\tilde{t} = f^N t \text{ in } R_f$$

gluing property:

Consider case $U = \operatorname{Spec} R$ (instead of R_f)

$U_i = X_{g_i}$ note: $U_i \cap U_j = X_{g_i g_j}$

given $s_i \in R_{g_i} = \mathcal{O}_X(X_{g_i}) = \mathcal{O}_X(U_i)$

s.t. $\underbrace{s_i|_{U_i \cap U_j}}_{\text{im in } R_{g_i g_j}} = s_j|_{U_i \cap U_j}$ want $\exists s \in \mathcal{O}_X(U)$ $s|_{U_i} = s_i$.

(Reduce to finite case)

$\forall i$, $g_i^{N_i} s_i = \text{im of } t_i \in R$ choose $N \geq N_i$ all i.

find $t_i \in R$ s.t. $t_i/1 = g_i^N s_i$

$s_i|_{ij} = s_j|_{ij}$ $\quad s_i = \dfrac{t_i}{g_i^N} = \dfrac{t_j}{g_j^N} = s_j$ in $R_{g_i g_j}$

$g_j^N t_i = g_i^N t_j$ in $R_{g_i g_j}$

so $(g_i g_j)^M g_j^N t_i = (g_i g_j)^M g_i^N t_j$ in R

or $g_j^{M+N} (g_i^M t_i) = g_i^{M+N} (g_j^M t_j)$ in R

what were the t_i's? we chose them so

$$t_i/1 = g_i^N s_i \text{ in } R_{g_i}$$

but we have then $g_i^M t_i/1 = g_i^{M+N} s_i$

so change $t_i \rightsquigarrow g_i^M t_i$ and

$N \rightsquigarrow N+M$

WLOG we have $g_j^N t_i = g_i^N t_j$ in R.

Now, the X_{g_i}'s cover, so $(g_i) = R \iff (g_i^N) = 1$

$\Rightarrow 1 = \sum a_i g_i^N$. Set $s = \sum a_i t_i$

then $s g_j^N = \sum_i a_i t_i g_j^N = \sum_i a_i t_j g_i^N$

$= t_j \sum a_i g_i^N = t_j$

so in R_{g_j}, $s g_j^N = t_j/1 = g_j^N s_j$

so $s/1 = s_j$ as desired $\square$.