Last time:

Showed that $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf of rings on Spec R

$\Rightarrow (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a ringed space.

(Moral lesson: sheaf property wasn't "formal" required work
Q: is there another perspective which would make this
more "obvious")

Stalks of $\mathcal{O}_{\operatorname{Spec} R}$: Let $\mathfrak{p} \in \operatorname{Spec} R$

$$\mathcal{O}_{\operatorname{Spec} R, \mathfrak{p}} = \varinjlim_{U \ni \mathfrak{p}} \mathcal{O}_{\operatorname{Spec} R}(U) = \varinjlim_{X_f \ni \mathfrak{p}} \mathcal{O}_X(X_f)$$

$X = \operatorname{Spec} R$

$$= \varinjlim_{f \notin \mathfrak{p}} R_f = R[(R \setminus \mathfrak{p})^{-1}]$$
$$= R_{\mathfrak{p}}$$

Stalks are local rings: $(X, \mathcal{O}_X)$ is a locally ringed space.

Locally ringed spaces $(X, \mathcal{O}_X)$ loc. ringed space

(i.e. $\mathcal{O}_{X,p}$ loc. ring all $p \in X$)

$m_{X,p}$ = max'l ideal of $\mathcal{O}_{X,p}$

Idea: $m_{X,p} \longleftrightarrow$ functions which vanish at p

$\mathcal{O}_{X,p} \longleftrightarrow$ functions which are regular on some open neighborhood of p

i.e. if $f \in \mathcal{O}_{X,P}$ didn't vanish at P, it would not vanish in a nbhd of P $\Rightarrow$ should be inv on some nbhd of P

$\Rightarrow$ should be inv. at P

$\Rightarrow f \notin m_P$

ex: $X = \mathbb{C}$ complex plane standard top

$\mathcal{O}_X(U)$ = holomorphic fcns $U \to \mathbb{C}$

$\mathcal{O}_{X,P} = \left\{ \sum_{i=0}^{\infty} a_i (z-P)^i \mid \text{conv in some disk about } P \right\}$

$m_{X,P} = \left\{ \sum_{i=1}^{\infty} a_i (z-P)^i \mid \text{conv in some disk about } P \right\}$

$\mathcal{O}_X(U) \to \mathcal{O}_{X,P}/m_{X,P} \cong \mathbb{C}$

$f \mapsto f(0)$

$\left[\Gamma : \text{LRS} \to \text{Rings} \right.$ $\rightsquigarrow$ easier pf of $\mathcal{O}_X$ a shf? $\left. \right]$

$\leftarrow$: Spec

Suppose $f: X \to Y$ cont. map of top spaces, have functors

$$\mathrm{Shv}(X) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Shv}(Y)$$

Defined by: $(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$

$(f^{pre-1}\mathcal{G})(V) = \varinjlim_{U \supset f(V)} \mathcal{F}(U)$ $\qquad$ $f^{pre-1}\mathcal{G}$ is a presheaf

$f^{-1}\mathcal{G} \equiv \widehat{(f^{pre-1}\mathcal{G})}$ sheafification.

Ex: H II.1.18 : $\mathrm{Hom}_{Shv_X}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Shv_Y}(\mathcal{G}, f_*\mathcal{F})$

i.e. $f^{-1} \dashv f_*$

<u>ex:</u> $X = pt$ $\quad$ Y = anything $\quad$ $f : \{p\} \longrightarrow X$

$f^{-1}\mathcal{F}$ = a sheaf on $\{p\}$ $\qquad$ $Shv_{\{p\}} \simeq \underline{Set}$

$C\text{-}Shv_{\{p\}} = C$

$f^{-1}\mathcal{F} = \mathcal{F}_p$

$f_*\mathcal{G}$ = "skyscraper sheaf" $\quad$ $f_*G(U) = \begin{cases} * & \text{if } p \notin U \\ G & \text{else} \end{cases}$

$\parallel$

G set.

<u>ex:</u> X = anything $\quad$ $Y = pt$ $\qquad$ $f : X \longrightarrow *$

$f_*\mathcal{F} = \mathcal{F}(X) = \Gamma(\mathcal{F}, X)$ global sections

$f^{-1}\mathcal{G} = (f^{-1}G)$ = loc. constant sheaf w/ value G

$f^{pre-1}G(X) = G$

$g_1, g_2 \in G$

consider section $\tilde{g}_1 \in (f^{-1}G)(U_1)$
value g_1

$\tilde{g}_2 \in f^{-1}G(U_2)$ value g_2

these agree on $U_1 \cap U_2$ values $\&$
need to sheafify to glue these sections together.

<u>ex:</u> $U \xhookrightarrow{\imath} X$ open inclusion.

$$\imath^{-1}\mathcal{F} = \mathcal{F}|_U \qquad \imath^{-1}(\mathcal{F})(V) = \varinjlim_{W \supset \imath(V)} \mathcal{F}(W) = \mathcal{F}(\imath(V))$$
$$= \mathcal{F}(V)$$

$V \subset U$

(end sheaf interlude)

<u>Def</u> A morphism of ringed spaces

$(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$

where $f: X \longrightarrow Y$ cont. map & $f^{\#}: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$

(or $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$)

i.e. $f^{\#}(U): \mathcal{O}_Y(U) \longrightarrow (f_*\mathcal{O}_X)(U)$
$$\| $$
$$\mathcal{O}_X(f^{-1}(U))$$

<u>idea:</u> given a function on U, $f^{\#}$ lets us "pull it back"

to get a function on $f^{-1}(U)$

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, dashed, "f^*\lambda \text{ ``=" } \lambda\circ f"] \arrow[dr, "f"'] & & \text{Values} \\ & U \arrow[ur, "\lambda"'] & \end{tikzcd}$$

<u>exercise</u> (suggested) : Given sheaves $\mathcal{F}, \mathcal{G}$ on X a top space,

for $U \subset X$ define $\mathcal{H}om_X(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{Sh_U}(\mathcal{F}|_U, \mathcal{G}|_U)$

i.e. $\mathcal{H}om_X(\mathcal{F},\mathcal{G}) : \mathrm{Open}(X)^{op} \to \mathrm{Sets}$

Show: $\mathcal{H}om_X(\mathcal{F},\mathcal{G})$ is a sheaf.

<u>exercise</u>: Given $(X,\mathcal{O}_X)$, $(Y,\mathcal{O}_Y)$ Ringed spaces,

$U \subset X$ set $\mathcal{H}(U) = \mathrm{Hom}_{RS}\big((U,\mathcal{O}_X|_U), (Y,\mathcal{O}_Y)\big)$

Show $\mathcal{H}$ is a sheaf.

Locally Ringed Spaces

if $\varphi : R \longrightarrow S$ map of comm. local rings

then $\varphi(r) \notin S^{*} \Rightarrow r \notin R^{*}$

$\varphi(r) \in m_S \Rightarrow r \in m_R$ i.e. $\varphi^{-1}(m_S) \subset m_R$

If $(X, \mathcal{O}_X) \xrightarrow{(f, f^{\#})} (Y, \mathcal{O}_Y)$

$\left(X \xrightarrow{f} Y \right)$ map of ringed spaces

w/ $p \in X$ we have: $f^{\#} : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$

$$(f^{\#})_{f(p)} : \mathcal{O}_{Y, f(p)} \longrightarrow (f_* \mathcal{O}_X)_{f(p)}$$

$$= \varinjlim_{U \ni f(p)} (f_* \mathcal{O}_X)(U) \qquad (f_* \mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$$

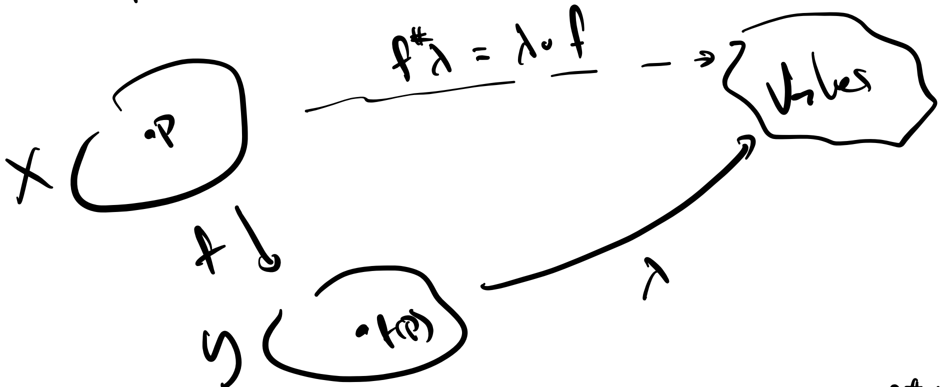
$$= \varinjlim_{\substack{p \in V \\ V = f^{-1}(U)}} \mathcal{O}_X(V)$$

$$\downarrow$$

$$\mathcal{O}_{Y, f(p)} \xrightarrow{f^{\#}_p} \mathcal{O}_{X, p} = \varinjlim_{p \in V} \mathcal{O}_X(V)$$

if this is really "pullback" we'd expect that

if pullback $f_p^\# \lambda$ vanishes at p $\Rightarrow$ λ vanishes at $f(p)$



and conversely, if λ vanishes at $f(p)$ then $f^\# \lambda$ vanish at p

i.e. $f_p^\#(m_{Y,f(p)}) \subset m_{X,p}$

for arb. LRSpaces

$X = 1pt = Y$ $\qquad$ $\mathcal{O}_Y \longleftrightarrow R$ loc. domain $\qquad \mathbb{C}((t))$

$\qquad\qquad\qquad\quad \mathcal{O}_X \longleftrightarrow \mathrm{frac}(R)$ $\qquad \mathbb{C}[[t]]$

$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ $\qquad$ $f^\#(r) \longrightarrow$ invertible

$R \xrightarrow[\text{inclusion}]{} \mathrm{frac}\, R$ $\qquad$ $r \neq 0$ $\quad f^\#(m_{Y,*}) \subset m_{X,*} = 0$

Problem: $r \in m_R$ are non-geometrically non-invertible!

<u>Def</u> $\varphi : R \to S$ lrings is a local homomorphism if

$\varphi(m_R) \subset m_S$ $(\Leftrightarrow m_R \subset \varphi^{-1}(m_S) \Leftrightarrow \varphi^{-1}(m_S) = m_R)$

Def $f : X \to Y$ LRS is a morphism of LRS if $\forall p \in X$, $f^\#_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism.

Prop (H II.2.3): $\mathrm{Hom}_{LRS}((\mathrm{Spec}\, B, \mathcal{O}_{\mathrm{Spec}\, B}), (\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A}))$

$$= \mathrm{Hom}_{Rng}(A, B)$$

Recall: Had $Spc_R = \mathrm{Fun}(R\text{-alg}, \underline{Set})$

$$(R\text{-alg})^{op} \xrightarrow{h} Spc_R$$
$$A \longmapsto \mathrm{Hom}(A, -) = h_A$$

Called the essential image of this functor the "Affine spaces"

h (Yoneda) is a fully faithful embedding - i.e.

$$\mathrm{AffSpc} \underset{eq}{\simeq} (R\text{-alg})^{op}$$

$F : \mathcal{C} \to \mathcal{D}$ functor

ess im(F) = full subcat of $\mathcal{D}$ whose objects are $\{d \in \mathrm{Ob}(\mathcal{D}) \mid d \cong Fc \text{ some } c \in \mathrm{Ob}(\mathcal{C})\}$

So: get an equiv. of cats

$$\mathrm{AffSpc} \longleftrightarrow \text{LRS of the form } (\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$$

A an R-alg.

oops - need morphisms to respect R...

Def A scheme is a locally ringed space $(X, \mathcal{O}_X)$ s.t. $\exists$ open cover $\{U_i\}$ of X s.t. $(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$

some rings A_i, $\cong$ is an iso of LRS.

$\boxed{\text{Next}}$ → prop of schemes & morphisms between them
→ gluing schemes (particular as functors)

$$
\begin{tikzcd}
\text{Affine Schemes} \arrow[r, leftrightarrow] \arrow[d, hook] & \text{Aff Spc} \arrow[d, hook] \arrow[dr, hook] & \\
\text{Schemes} \arrow[r, leftrightarrow] & (\qquad) \arrow[r] & \text{Spc}
\end{tikzcd}
$$

Remark: $\underline{\text{Affine Sch}} \simeq (\text{Comm. Rngs})^{op} \simeq \underline{\text{Aff Spc}} \hookrightarrow \underline{\text{Spc}}$

given an affine scheme $X = \operatorname{Spec} A$

X (& A) defined by the functor

$$
\begin{aligned}
S_X : \underline{\text{Com Rng}} &\longrightarrow \underline{\text{Sets}} \\
B &\longmapsto \operatorname{Hom}_{\text{rngs}}(A, B) \\
&\qquad = \operatorname{Hom}_{\text{LRS}}(\operatorname{Spec} B, \operatorname{Spec} A)
\end{aligned}
$$

schemes

more generally, if X is any scheme.

Can still define S_X by same formula

$$\begin{array}{ccc} S_X : \underline{\text{CommRings}} & \longrightarrow & \underline{\text{Sets}} \\ (\text{Affine})^{op} \cong \quad B & \longmapsto & \text{Hom}_{Sch}(\text{Spec } B, X) = X(B) \\ \big\downarrow & & \\ (Sch)^{op} & \longrightarrow & \text{Sets} \\ Y & \longmapsto & \text{Hom}_{Sch}(Y, X) \end{array}$$

Fact: $S_X : \underline{\text{CommRings}} \longrightarrow \underline{\text{Sets}}$ still determines X.

$$Sch \hookrightarrow Spc_{Zar}$$
$$AffSch \longrightarrow Spc_{Zar}, \quad h_A$$

Guess: 1) ess image of consists of Zariski sheaves which are colimits in sheaves of Affine ones?

2) Zariski sheaves S s.t. $\exists$ affine U & $S_U \twoheadrightarrow S$ sheaf surjective map & (locally) open map.