Fiber Products

Def Given a diagram of maps of sets

$$\begin{tikzcd} & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & S \end{tikzcd}$$

$f_1^{-1}(s) \times f_2^{-1}(s)$ $\quad f_2^{-1}(s) \quad f_1^{-1}(s) \quad s$

the fiber product $X_1 \times_S X_2$ is the set

$X_1 \times_S X_2 = \{(x_1, x_2) \mid f_1 x_1 = f_2 x_2\}$

This gives "universal diagram"

$$\begin{tikzcd} X_1 \times_S X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & S \end{tikzcd}$$

i.e. $\forall\, Y$, given

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & S \end{tikzcd}$$

commutes

$\exists!\; Y \to X_1 \times_S X_2$ s.t. diagram commutes

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] \arrow[dd] & & X_2 \arrow[dd] \\ & X_1 \times_S X_2 \arrow[ur] \arrow[dl] & \\ X_1 \arrow[rr] & & S \end{tikzcd}$$

i.e. $\mathrm{Hom}(Y, X_1 \times_S X_2) = \mathrm{Hom}(Y, X_1) \times_{\mathrm{Hom}(Y,S)} \mathrm{Hom}(Y, Y_2)$

(these are all sets still)

This gives the general def of the fiber product:

Given
$$\begin{tikzcd} X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & S & \end{tikzcd}$$
in a category $\mathcal{C}$, we get

the functor $Y \longmapsto Hom(Y, X_1) \times_{Hom(Y,S)} Hom(Y, X_2)$

$\mathcal{C}^{op} \longrightarrow Sets$

we say $X_1 \times_S X_2$ exists if it represents this functor.

i.e. $Hom(Y, X_1 \times_S X_2) \simeq Hom(Y, X_1) \times_{Hom(Y,S)} Hom(Y, X_2)$

<u>Remark</u>: If $F_1, F_2, S : \mathcal{C}^{op} \longrightarrow Sets$

are functors (presheaves)

then can check $F_1 \times_S F_2$ exists in $Fun(\mathcal{C}^{op}, Sets)$

and are defined as $(F_1 \times_S F_2)(x) = F_1(x) \times_{S(x)} F_2(x)$

- Further, if $\mathcal{C}^{op} = Rings$ (Sch^{op}) then if F_1, F_2, S are Zariski sheaves, $\underset{Fun(Rings, Sets)}{\cap}$ $Fun(Sch^{op}, Sets)$

so is $F_1 \times_S F_2$

- For any cat, if $X_1 \times_S X_2$ exists in $\mathcal{C}$, and h_{X_1}, h_S, h_{X_2} are corresp. rep functors in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$

  Then $h_{(X_1 \times_S X_2)} = h_{X_1} \times_{h_S} h_{X_2}$

  i.e. $h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$ preserves fibr products.

<u>Example</u>: $\mathcal{C}$ = schemes, X_1, X_2, S are affine schemes

$X_1 = \mathrm{Spec}\, A_1$ $\quad X_2 = \mathrm{Spec}\, A_2$ $\quad S = \mathrm{Spec}\, B$

what is $X_1 \times_S X_2$ determined by its associated Zariski shf.

$\mathcal{S}_{X_1 \times_S X_2} = h_{X_1 \times_S X_2}$ ← scheme.

"Zariski shf" $h_{X_1 \times_S X_2}(R) = \mathrm{Hom}_{sch}(\mathrm{Spec}\, R, X_1 \times_S X_2)$

ring R

$h_{X_1 \times_S X_2}(R) = \mathrm{Hom}_{sch}(\mathrm{Spec}\, R, X_1 \times_S X_2)$

$= \mathrm{Hom}_{sch}(\mathrm{Spec}\, R, X_1) \times_{\mathrm{Hom}_{sch}(\mathrm{Spec}\, R, S)} \mathrm{Hom}_{sch}(\mathrm{Spec}\, R, X_2)$

$= \mathrm{Hom}_{rng}(A_1, R) \times_{\mathrm{Hom}_{rng}(B, R)} \mathrm{Hom}_{rng}(A_2, R)$

Write $\widetilde{X_1 \times_S X_2}$ to be functor

$$Y \longmapsto \mathrm{Hom}_{Sch}(Y, X_1) \times_{\mathrm{Hom}_{Sch}(Y,S)} \mathrm{Hom}(Y, X_2)$$

this is the fiber product $h_{X_1} \times_{h_S} h_{X_2}$ in $\mathrm{Fun}(Sch^{op}, Set)$

or equiv. in ZarShv

we want to show that $\widetilde{X_1 \times_S X_2}$ is representable by a scheme. i.e. $h_Z \simeq \widetilde{X_1 \times_S X_2}$

We compute:

$$\widetilde{X_1 \times_S X_2}(R) = (h_{X_1} \times_{h_S} h_{X_2})(R)$$

$$= h_{X_1}(R) \times_{h_S(R)} h_{X_2}(R)$$

$$= \mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, X_1) \times_{\mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, S)} \mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, X_2)$$

$$= \mathrm{Hom}_{rg}(A_1, R) \times_{\mathrm{Hom}_{rgs}(B, R)} \mathrm{Hom}_{rgs}(A_2, R)$$

$$= \mathrm{Hom}(A_1 \sqcup_B A_2, R) = \mathrm{Hom}(A_1 \otimes_B A_2, R)$$

pushout

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A_2 \\ A_1 & \end{tikzcd}$$

$$\widetilde{X_1 \times_S X_2}(R) = \mathrm{Hom}_{\mathrm{rgs}}(A_1 \otimes_B A_2, R) = \mathrm{Hom}_{\mathrm{sch}}(\mathrm{Spec}\, R, \mathrm{Spec}(\,))$$

$$= h_{\mathrm{Spec}(A_1 \otimes_B A_2)}(R)$$

get an $\simeq$ of functors $\widetilde{X_1 \times_S X_2} \simeq h_{\mathrm{Spec}\, A_1 \otimes_B A_2}$

(Xiangrui: $\mathrm{Spec} : \mathrm{Rgs} \to \mathrm{LRS}$ is right adj to Γ, it preserves limits $\Rightarrow$ can compute fib x's of affine schs via pushouts in rgs!)

Summary of general fiber product situation

Construct $X_1 \times_S X_2$ for schemes
$$\begin{tikzcd} X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & S & \end{tikzcd}$$
gluing.

e.g. choose W_i's cover of S then $X_1 \times_S X_2$ will be obtained by gluing $f_1^{-1}(W_i) \times_{W_i} f_2^{-1}(W_i)$

i.e. consider functors

$$\begin{tikzcd} & \widetilde{f_1^{-1}(W_j) \times_{W_j} f_2^{-1}(W_j)} \\ \widetilde{f_1^{-1}(W_i) \times_{W_i} f_2^{-1}(W_i)} & \widetilde{f_1^{-1}(W_i \cap W_j) \times_{W_i \cap W_j} f_2^{-1}(W_i \cap W_j)} \arrow[l] \arrow[u] \end{tikzcd}$$

So WLOG, can assume $S = \operatorname{Spec} B$ is affine.

Similarly, if $\{U_i\}$ cover X_1 $\{V_j\}$ cover X_2

then can obtain $X_1 \times_S X_2$ by ~~by~~ gluing $U_i \times_S V_j$

if these all exist.

So can reduce to all affine, done.

Def for $X \xrightarrow{\pi} Y$ and $Z \hookrightarrow Y$ (open or closed inclusion)

the (scheme-theoretic) inverse image of Z in X

is defined as $\pi^{-1}(Z) \equiv X \times_Y Z$

$$\begin{tikzcd} \pi^{-1}(Z) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ Z \arrow[r] & Y \end{tikzcd}$$

ex: if Y a scheme $\pi: X \to Y$, $y \in Y$ pt.

observe can construct a map

$\operatorname{Spec}(\operatorname{frac} \mathcal{O}_{Y,y}/m_y) \to \operatorname{Spec} \mathcal{O}_{Y,y}/m_y \to \operatorname{Spec} \mathcal{O}_{Y,y} \to Y$

via: for any $\operatorname{Spec} R \subset Y$ open affine containing y

$$\begin{array}{l} R \longrightarrow R_y \\ \quad\quad\quad \overset{\shortparallel}{\mathcal{O}_{y,y}} \searrow R_y/yR_y \quad \text{domain} \\ \quad\quad\quad\quad\quad\quad\quad\quad \downarrow \end{array}$$

y considered as a pt in Spec $R = U$

"residue field" of y $\quad \kappa(y) = \text{frac}(R_y/yR_y)$

<u>Back to reality for a moment</u>

points & residue fields of schemes

R = base $\quad A$ an R-alg. $\quad$ Spec A

$$A = R[x_1, \ldots, x_n]/(f_i) \quad = \frac{\mathbb{Z}[x,y]}{x^2+y^2} = A$$

$$\mathfrak{p} = (x,y) \qquad A/\mathfrak{p} \simeq \mathbb{Z}$$

$$\frac{\mathbb{Z}[x,y]}{x^2+y^2} \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q}$$

$$x \longmapsto 0$$

$$y \longmapsto 0$$

$$\mathfrak{p} = (x,y,3) \qquad \longrightarrow \mathbb{Z}/3\mathbb{Z}$$

$$A \longrightarrow A/\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$$

$$\frac{\mathbb{Z}[x,y]}{x^2+y^2} \longrightarrow \mathbb{Z}[i] \hookrightarrow \mathbb{Q}(i) \hookrightarrow \mathbb{C}$$

$$x \longmapsto i$$

$$y \longmapsto 1$$

$$\mathfrak{p} = (x^2+1, y-1)$$

if $x \in X$ point. say $\operatorname{Spec} A \subset X$ open affine containing x

i.e. $x \leftrightarrow \mathfrak{p} \triangleleft A$ $\quad \underbrace{A \to A/\mathfrak{p} \to \operatorname{frac} A/\mathfrak{p}}_{\text{encodes a soln to eqns describing } A \text{ over the field } \kappa(x).} = \text{field } \kappa(x)$

$$\operatorname{Spec} \kappa(x) \longrightarrow \operatorname{Spec} A \subset X$$

scheme w/ 1 pt as a top space. $\quad pt \longmapsto x$

Remark: a morphism $\operatorname{Spec} L \to X$ $\quad X$ a scheme

is interpreted as a solution to equs defining X in the field L.

if x is the image of the single point in $\operatorname{Spec} L$

then we get $\operatorname{Spec} L \to \operatorname{Spec} \kappa(x) \to X$ uniquely factors

$A \to L$ kernel is prime.

$A/\ker \hookrightarrow L$ domain.

$\mathfrak{p}$ = kernel

$$\begin{tikzcd} A/\mathfrak{p} \arrow[r] \arrow[dr] & L \\ & \mathrm{frac}(A/\mathfrak{p}) \arrow[u, hook] \end{tikzcd}$$

i.e. residue field is "smallest field yielding a soln corresp. to point x" "field of definition of x"

<u>**Def**</u> A geometric point of a scheme X is a morphism $\operatorname{Spec}\Omega \to X$, Ω = alg. closed field.

<u>note:</u> for any point $x \in X$ $\exists$ (many) geom. pts

$$\operatorname{Spec}\Omega \to \operatorname{Spec}\kappa(x) \to X$$

$\uparrow$ constructed above.

$$A \to \mathrm{frac}\, A/\mathfrak{p} \hookrightarrow \Omega$$

Spec L has 1 pt (0)

L field

$p \in X$

$$\begin{tikzcd} \operatorname{Spec} F \arrow[r] & X \\ pt \arrow[r, dashed] & p \end{tikzcd}$$

$$\begin{tikzcd} & F & A \ni p \arrow[l, "f"'] \arrow[d, bend left] \\ \kappa(\mathfrak{p}) = \mathrm{frac}(A/\mathfrak{p}) \arrow[ur] & & A/\mathfrak{p} \arrow[ll, bend left] \arrow[ul, bend left] \end{tikzcd}$$

$f^{-1}(0) = p$

$\mathbb{Z}[x]$ $\qquad$ $\mathbb{F}_p$

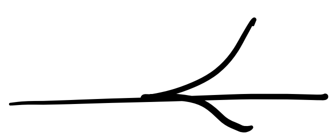
$\mathbb{Z}[x] \longrightarrow \dfrac{\mathbb{Q}[x]}{f}$

$\mathbb{F}_p[x]$

<u>Fiber product</u>

<u>Def</u> for $Z_1, Z_2 \hookrightarrow X$ the scheme-theoretic intersection is $Z_1 \cap Z_2 = Z_1 \times_X Z_2$

<u>ex:</u> $Z(y^2-x^3) \cap Z(y)$ in $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$



represented by

$$\frac{k[x,y]}{y^2-x^3} \otimes_{k[x,y]} \frac{k[x,y]}{y}$$

$$= \frac{k[x,y]}{(y,y^2-x^3)} = \frac{k[x]}{x^3}$$

mult. 3 because $3 = \dim_k k[x]/x^3$

$\Big\{$ $\operatorname{Spec} k[x]/x^3$

points $\leftrightarrow$ primes of $\frac{k[x]}{x} = k$

Dictionary :

X a scheme   "point"

scheme theoretic pt = pt in underlying top space of X

an L-point (L a field) = element of $\mathrm{Hom}(\mathrm{Spec}\,L, X)$

a geometric pt := Ω-pt where $\Omega = \overline{\Omega}$ field.