# Combinatorics, supplementary

Daniel Krashen

September 28, 2016

# Contents

# Chapter 1

# Distribution problems and multisets

## 1.1 Multisets

### 1.1.1 Language

Let's start with some examples. Here are some sets with elements taken from $1, 2, 3$:

$$\{1,2\} \qquad \{1,3\} \qquad \{1\} \qquad \{1,2,3\}$$

and here are some multisets with elements taken from $1, 2, 3$:

$$\{1,1,2\} \qquad \{3\} \qquad \{1,2,2,3,3,3,3,3,3,3\}$$

all of these are generally referred to as "multi-subsets" of the set $\{1, 2, 3\}$.

To move towards a precise definition, consider an alternative way of representing the multisets above. Let's try the informal notation

$$\{1,1,2\} = \{1(2 \text{ times}), 2(1 \text{ time})\}$$

or

$$\{1,2,2,3,3,3,3,3,3,3\} = \{1(1 \text{ time}), 2(2 \text{ times}), 3(7 \text{ times})\}$$

In this way, it makes sense to think of a multiset as a set, together with an assignment of a "multiplicity" to each object (how many times the object occurs).

We can therefore formally define

**Definition 1.1.1.** *A multiset is a pair $(S, m)$ consisting of a set $S$ and a function $m : S \to \mathbb{Z}_{>0}$.*

For example, the multiset $\{1, 1, 2\}$ we think of as the pair $(\{1, 2\}, m)$, where $m(1) = 2$ and $m(2) = 1$. Since notationally it is easier to write $\{1, 1, 2\}$, we will generally stick to this notation for multisets.

Note that just as for sets, the order in which we write our elements doesn't matter. That is to say, $\{1, 1, 2\} = \{1, 2, 1\} = \{2, 1, 1\}$.

### 1.1.2 Counting

Let's count the number of multisets having 10 elements, taken from the set $\{1, 2, 3\}$. For example, here are a few:

$$\{1,1,1,2,2,2,2,3,3,3\}, \{1,1,1,1,1,1,1,1,1,1\}, \{1,1,2,3,3,3,3,3,3,3\}$$

How do we count these? Here's a simple trick: if we write the numbers which occur within the multiset in order (as we have done above), We can easily fill in which numbers are which, if we know the "transition points in the sequence." That is, if we imagine putting "separators" into our sequences above, to yield:

$$(111s2222s333), (1111111111ss), (11s2s33333333)$$

(where $s$ stands for "separator"). To be clear, the rule here is:

1. the items to the left of all separators are 1's
2. the items between the separators are 2's
3. the items to the right of all separators are 3's

Note that with these rules, we really didn't need to specify our numbers at all. That is to say, if $n$ stands for "number," we could represent the above sequences as:

$$(nnnsnnnnsnnn), (nnnnnnnnnnss), (nnsnsnnnnnnn)$$

and we can easily fill in the numbers as above. This tells us that what we are really doing is choosing which 2 or our 12 symbols should be $s$. Therefore, the number of ways of choosing these are $\binom{12}{2}$.

## 1.2 Partition-type distribution problems

### 1.2.1 Language

In this section, the basic building blocks of our concepts are subsets and permutations. Given a set $S$, at this point, we're all pretty familiar with subsets of $S$, but recall:

**Definition 1.2.1.** *A permutation of length $k$ of a set $S$ is a sequence of $k$ distinct elements of $S$. That is, a sequence of the form $(s_1, \ldots, s_k)$.*

Recall that the number of such permutations is given as the falling factorial $n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$, where $n = \#S$.

In this section, we will consider the problem of breaking a set $S$ up into some number of parts. We will always assume that these parts are nonoverlapping, and that together they account for all the elements of $S$. Our different problems will arise based on

- whether or not we order the elements in the individual parts,
- whether or not we order the parts themselves
- whether or not we fix the number of elements within each part

We can think of this as follows: think of the elements of $S$ as distinct (labelled) balls which we are distributing into some number of bins. If the bins are ordered/labelled, then consider placing balls 1 and 2 in bin 1 and balls 3 and 4 into bin 2 as different then placing 1 and 2 into bin 2 and placing balls 3 and 4 into bin 1. If the bins are not labelled, we consider these the same distribution. In the case where the elements are ordered within the parts, we visualize the bins as tubes, wherein we have to specify the order in which the balls are placed within the bins. Again, we have the same variations based on whether or not the bins themselves are labelled.

Courtesy of Chen Chen, we have the following break-down of counting problems, encapsulating these possibilities:

| | |
|---|---|
| sequences of subsequences of $S$ | sets of subsequences of $S$ |
| sequences of subsets of $S$ | sets of subsets of $S$ |

For example,

- in the upper left, the sequence of sequences $((1, 2), (3, 4))$ describes the placement of balls into labelled tubes with a given order,

- for the lower left, the sequence of sets $(\{1,2\},\{3,4\})$ represents placing balls into labelled bins (order doesn't matter within a bin),
- for the upper right, the set of sequences $\{(1,2),(3,4)\}$ represents placing the balls in the specified order into two bins, which are not themselves labelled,
- for the lower right, the set of sets $\{\{1,2\},\{3,4\}\}$ represents placing the balls into two unlabelled bins. Here, neither the order of the balls, nor the order of the bins matters.

### 1.2.2 Sequences of subsets: ordered partitions and multinomial coefficients

**Definition 1.2.2.** *Let $S$ be a set. An **ordered partition** of $S$ is a sequence $(S_1, S_2, \ldots, S_n)$ of pairwise disjoint subsets of $S$ whose union is all of $S$. For a given ordered partition $(S_1, \ldots, S_n)$, we define the **type** of the type of the partition to be the sequence $(\#S_1, \ldots, \#S_n)$ of sizes of the sets $S_i$.*

For example, if $S = \{1,2,3,4,5,6,7\}$, then the sequence $(\{2,3,4\},\{1,5\},\{6,7\})$. is an ordered partition of $S$ of type $(3,2,2)$.

Note that this is the same ordered partition as $(\{2,4,3\},\{5,1\},\{7,6\})$, since the order in which we write elements of a set dosen't matter. On the other hand $(\{2,3,4\},\{6,7\},\{1,5\})$ is a different ordered partition, since the order in which the elements of a sequence are placed matters.

**How to count them** The number of these, for a given type, is the multinomial coefficient. That is:

$$\binom{k}{m_1, m_2, \ldots, m_n} = \#\left\{\begin{array}{c}\text{ordered partitions of} \\ \{1,\ldots,k\} \text{ of type } (m_1,\ldots,m_n).\end{array}\right\}$$

One can also ask for the number of ways to distribute the $k$ elements into $n$ without specifying how many elements must be in a given set...

### 1.2.3 Sets of subsets: partitions and Sterling numbers

**Definition 1.2.3.** *Let $S$ be a set. A partition of $S$ is a collection of (nonempty) subsets $\{S_1, S_2, \ldots, S_n\}$ of $S$ such that $\cup_{i=1}^n S_i = S$ and the sets $S_i$ are mutually disjoint.*

**Definition 1.2.4.** *For a given partition $\{S_1, \ldots, S_n\}$ the orders of the parts of the partition give a multiset $\{\#S_1, \ldots, \#S_n\}$ called the "type" of the partition.*

**How to count them** The number of partitions of a set $S$ with $k$ elements into $n$ nonempty parts is the Sterling number of the second kind $S(k,n)$.

### 1.2.4 Sequences of sequences: broken permutations and Lah numbers

**Definition 1.2.5.** *Let $S$ be a set. A **broken permutation** of $S$ is a set of permutations of $S$ of various lengths $\{P_1, P_2, \ldots P_n\}$ (that is to say, each $P_i$ is an ordered list of distinct elements of $S$), such that every elemnet of $S$ is in exactly one of the permutations $P_i$.*

**Definition 1.2.6.** *Given a broken permutation $\{P_1, \ldots, P_n\}$ the multiset of the lengths of the permutations $\{length(P_1), \ldots, length(P_n)\}$ is called the **type** of the broken permutation.*

**How to count them** The number of broken permutations of a set $S$ with $k$ elements into $n$ nonempty parts is the Lah number $L(k,n)$.

### 1.2.5 Sequences of sequences: ordered broken permutations

**Definition 1.2.7.** *Let $S$ be a set. An **ordered broken permutation** of $S$ is a sequence of permutations of $S$ of various lengths $(P_1, P_2, \ldots P_n)$, such that every elemnet of $S$ is in exactly one of the permutations $P_i$.*

**Definition 1.2.8.** *Given an ordered broken permutation $(P_1, \ldots, P_n)$, the sequence of the lengths of the permutations $(length(P_1), \ldots, length(P_n))$ is called the **type** of the ordered broken permutation.*

**How to count them** ...