Maximum Likelihood estimators

If we have pop distribution $f_\theta(x)$

if we observe $X = x_0$ we guess value for θ

which maximize $g(\theta) = f_\theta(x_0)$

Finite example

Five people take a test, estimate # pass (θ)

Pick at random 3 people: of them 2 passed, 1 failed

$\theta = \cancel{0}, \cancel{1}, 2, 3, 4, \cancel{5}$

$P(\text{observation} \mid \theta)$

$P(\quad \mid \theta = 0) = 0$

$P(\quad \mid \theta = 1) = 0$

$$P(\quad \mid \theta = 2) = \frac{\binom{2}{2}\binom{3}{1}}{\binom{5}{3}} = \frac{1 \cdot 3}{10} = \frac{3}{10}$$

$$P(\quad \mid \theta = 3) = \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}} = \frac{3 \cdot 2}{10} = \frac{6}{10}$$

$$P(\quad \mid \theta = 4) = \frac{\binom{4}{2}\binom{1}{1}}{\binom{5}{3}} = \frac{6 \cdot 1}{10} = \frac{6}{10}$$

$P(\quad | \theta = 5) = 0$

not unique but most likely are $\theta = 3$
$\theta = 4$

Remarks about MLE

- don't need to be unique
- often unique for common distributions, cont. case
- with certain assumptions, they are consistent.
- min'l variance.

Don't have to be unbiased

have invariance property:

if $\hat{\Theta}$ a MLE for θ then
$g(\hat{\Theta})$ is a MLE for $g(\theta)$
if g cont.

Some reasonable method for consistent low variance estimators.

Example Normal population $\sigma^2 = 3$ μ unknown

$$f_\mu(x) = \frac{1}{\sqrt{3 \cdot 2\pi}} e^{-\frac{1}{6}(x-\mu)^2}$$

If we observe $n = 12$, $\bar{X} = 15$ what is a MLE for μ.

which value of μ maximizes

$f_\mu(x_1, \ldots, x_n)$ if $\bar{x} = 15$?

$$f_\mu(\vec{x}) = \frac{1}{\sqrt{6\pi}} e^{-\frac{1}{6}\sum_{i=1}^{12}(x_i-\mu)^2}$$

Usual trick:

$$\sum (x_i-\mu)^2 = \sum_i \left((x_i - \bar{x}) + (\bar{x} - \mu) \right)^2$$

$$= \sum (x_i - \bar{x})^2 + \sum_i (\bar{x}-\mu)^2$$
$$+ 2 \sum_i (x_i - \bar{x})(\bar{x} - \mu)$$

$$= \sum (x_i - \bar{x})^2 + n(\bar{x}-\mu)^2$$
$$+ 2(\bar{x}-\mu)\underbrace{\left(\sum (x_i - \bar{x})\right)}_{0}$$

$$\sum (x_i - \bar{x}) = \sum x_i - \sum \bar{x} = n\bar{x} - n\bar{x} = 0$$

$$f_\mu(\vec{x}) = \frac{1}{\sqrt{6\pi}}\, e^{-\frac{1}{6}\sum_{i=1}^{12}(x_i-\mu)^2}$$

$$= \frac{1}{\sqrt{6\pi}}\, e^{-\frac{1}{6}\sum_{i=1}^{12}(x_i-\bar{x})^2} \cdot e^{-\frac{12}{6}(\bar{x}-\mu)^2}$$

choose μ to maximize this

$$\Rightarrow \text{maximize } e^{-\frac{12}{6}(\bar{x}-\mu)^2}$$

max at $\bar{x}=\mu$.

MLE estimator for μ is $\bar{x}$

$$\boxed{\bar{X}}$$

<u>General case</u> both μ & σ^2 unknown

$$f_{\mu,\sigma^2}(\vec{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}\, e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2}$$

want to solve for μ,σ^2 maximize this we'll find max always is at $\mu = \overline{X}$

what what σ^2 ?

$\frac{\partial}{\partial \sigma^2} f_{\mu, \sigma^2}$ maxing $f_{\mu,\sigma^2} \Leftrightarrow$ maxing $\ln f_{\mu,\sigma^2}$

$$f_{\mu,\sigma^2}(\vec{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2}$$

$v = \sigma^2$

$$f_{\mu,v}(\vec{x}) = \left(\frac{1}{2\pi v}\right)^{n/2} e^{-\frac{1}{2v} \sum (x_i - \mu)^2}$$

$$\ln f_{\mu,v}(\vec{x}) = -\frac{n}{2}\left(\ln 2\pi + \ln v\right) - \frac{1}{2v} \sum (x_i - \mu)^2$$

$$\frac{\partial}{\partial v}\left(\quad \right) = -\frac{n}{2} \frac{1}{v} + \frac{1}{2v^2} \sum (x_i - \mu)^2 = 0$$

mult by $2v^2$

$$-nv + \sum (x_i - \mu)^2 = 0$$

$$v = \frac{\sum (x_i - \mu)^2}{n}$$

$\mu = \overline{X}$

MLE estimators are $\mu = \overline{X}$

$v = \dfrac{\sum_i (X_i - \overline{X})^2}{n}$

" σ^2

$\hat{\mu} = \overline{X}$

$\hat{\sigma}^2 = \dfrac{\sum_i (X_i - \overline{X})^2}{n}$

$\Rightarrow \hat{\sigma} = \sqrt{\dfrac{\sum_i (X_i - \overline{X})^2}{n}}$ is an MLE estimator for σ