Composite hypotheses

hypothesis: assertion concerning the dist. for a population

typically the value of some parameter(s)

$\theta = a \qquad \theta \geq a \qquad \theta < b \qquad \theta \neq c$

Common case: simple null hypothesis H_0 $(\theta = \theta_0)$

and a composite alternative H_1

Such a test is called a "test of significance"

α = "significance level"

common subcases: alt. hyp is of the form

$\underbrace{\theta > \theta_0 \ , \ \theta < \theta_0}_{\text{one sided alternates}} \qquad \underbrace{\text{or } \theta \neq \theta_0}_{\text{two sided alternates.}}$

"one tailed" or "two tailed" tests

Example: coin flip / Bernoulli variable $\theta = p(\text{heads})$

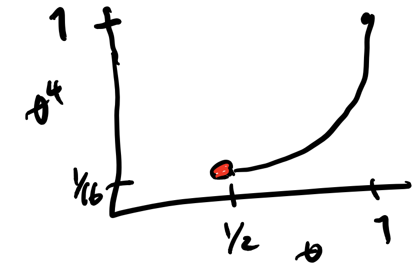
$H_0 : \theta = 1/2$ test: accept H_0 unless 4 heads.

$H_1 : \theta > 1/2$ if 4 heads $\rightarrow H_1$

$\alpha = P(\text{type } 1) = \frac{1}{16} \qquad P(\text{type } 2) = \text{undef.}$

$\pi(\theta) = 1 - P(\text{type } 2) = P(\text{accepting } H_1 \text{ if } H_1 \text{ is true})$

θ s.t. H_1 $= \theta^4$ $\quad \theta \in (\frac{1}{2}, 1]$



$\pi(\theta) = P(\text{accepting } H_1 \text{ given } \theta)$

Basic game: make $\pi(\theta)$ large while having $\pi(\theta) \le \alpha$ for θ satisfying H_0.

Def We say a critical region of size $\le \alpha$ is uniformly most powerful region (of size α) if $\pi(\theta)$ is $\geq$ power function for any other crit region of size $\le \alpha$.

Perverse example

$\alpha = \frac{1}{4}$ coin flip, 4 flips

$H_0 : \theta = \frac{1}{2}$

$H_1 : \theta > \frac{1}{2}$

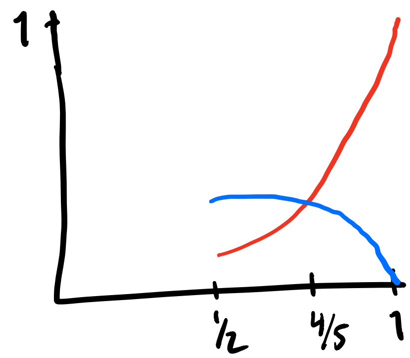
test 1: if exactly 3 heads $\Rightarrow H_1$
else H_0

test 2: if exactly 4 heads $\Rightarrow H_1$
... H_0

$P(\text{type 1 in test 1}) = 4\left(\frac{1}{2}\right)^4 = \frac{4}{16} = \frac{1}{4}$

$P(\text{type 1} \;\ldots\; 2) = \frac{1}{16}$

- $\pi_1(\theta) = P(3 \text{ heads} \mid \theta) = 4\theta^3(1-\theta)$
- $\pi_2(\theta) = P(4 \text{ heads} \mid \theta) = \theta^4$



for $\theta \in (\frac{1}{2}, 4/5)$ test 1 more powerful

for $\theta \in (4/5, 1)$ test 2 more powerful

Likelihood Ratio tests

recall: Neyman-Pearson:

test had form $\dfrac{f(x|H_0)}{f(x|H_1)} \le K$ in critical region

$\ge K$ outside

giving a statistic

$$\Lambda = \frac{f(X|H_0)}{f(X|H_1)}$$ statistic

test: $\lambda \geq K$ crit region.

Similar in composite case

replace $f(x \mid H_0)$ $\quad f(x \mid \theta)$

$f(x \mid H_1)$

$\max_{\theta \text{ s.t. } H_0} f(x;\theta) = \max_{\theta \text{ s.t. } H_0} L(\theta; x)$

$\max_{\theta \text{ s.t. } H_0 \text{ or } H_1} f(x;\theta) = \max_{\theta \text{ s.t. } H_0 \text{ or } H_1} L(\theta; x)$

$$\lambda_{top}(x) = \max_{\substack{\theta \text{ s.t.} \\ H_0}} L(\theta; x)$$

$$\lambda_{bottom}(x) = \max_{\substack{\theta \text{ s.t.} \\ H_0 \text{ or } H_1}} L(\theta; x)$$

randomness (functions of X)

$$\Lambda_{top} = \lambda_{top}(X) \qquad \Lambda_{bottom} = \lambda_{bottom}(X)$$

$$\Lambda = \frac{\Lambda_{top}}{\Lambda_{bottom}}$$ "likelihood ratio statistic"

measured values $\lambda \rightsquigarrow$ likelihood ratio test

$\boxed{\lambda \leq k \quad \text{crit region}}$

Example

normal population: known variance σ^2

unknown mean

$$\theta = \mu \qquad L(\mu; x) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i-\mu)^2}$$

$H_0: \theta = \mu = \mu_0$

$H_1: \mu \neq \mu_0$

$$\lambda_{num}(x) = \max_{\mu \text{ s.t. } H_0} L(\mu; x) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum (x_i-\mu_0)^2}$$

$$\lambda_{den}(x) = \max_{\mu \; H_0 \text{ or } H_1} L(\mu; x) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum (x_i-\bar{x})^2}$$

$$\lambda = \frac{\lambda_{num}}{\lambda_{den}} = \frac{e^{-\frac{1}{2\sigma^2}\sum_i (x_i-\mu_0)^2}}{e^{-\frac{1}{2\sigma^2}\sum (x_i-\bar{x})^2}} = e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}$$

test has form $\lambda \leq k$ crit region

$$e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2} \leq k$$

$$(\bar{x}-\mu_0)^2 \geq c \iff \boxed{|\bar{x}-\mu_0| \geq K}$$

to finish, find a region s.t.

$$P(|\bar{x}-\mu_0| \geq k \mid \mu=\mu_0) = \alpha$$

our estimator $\Lambda = e^{-\frac{n}{2\sigma^2}(\bar{X}-\mu_0)^2}$

$$= e^{-\frac{1}{2}\underbrace{\left(\underbrace{\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}}_{Z}\right)^2}_{\chi^2_1}}$$

<u>Theorem</u> under "general" hypotheses, likelihood ratio test for a two sided alt. $H_0: \theta=\theta_0$, $H_1: \theta \neq \theta_0$

gives an estimator Λ s.t.

$-2\ln\Lambda$ approaches a χ^2_1 distribution for large n! (i.e. cdf for $-2\ln\Lambda$ at any value x approaches val of χ^2_1 as $n\to\infty$)

"general" X_i iid $f(x_i;\theta)$

$\theta \neq \theta' \Rightarrow f$ diff. $\int f(x_i;\theta)\,dx$

given θ' $\exists\, \varepsilon$ s.t. $|\theta-\theta'|<\varepsilon$ then $\left|\frac{\partial^3}{\partial\theta^3}\log f(x_i;\theta)\right| < M(x)$

$$E_{\theta'}[M(X)] < \infty$$